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Optimality and Stability of a Distribution of Talent in N-team Leagues with Revenue Sharing and Salary Caps

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Abstract: This paper shows negative results for the following fundamental problems: (i) can the improvement of competitive balance lead to the maximization of the total league profits?, and (ii) can the optimal distribution of talent be achieved through a process of spontaneous player trading among teams under the regulatory systems used in the professional sports league of North America (e.g. revenue sharing and salary caps) when the initial distribution of talent is different from it?

Keywords: Optimality of a Talent Distribution; Stability of a Talent Distribution; Player Trading; Revenue Sharing; Salary Caps

1. Introduction

In the professional sports league of North America, various regulatory systems are used to improve competitive balance in the league. This is because since the "uncertainty of outcome hypothesis" was first articulated by Neal (1964), it has been widely believed that the improvement of competitive balance increases fan interest in games, each team's profits, and therefore the total league profits. The impact of regulatory systems on competitive balance has been studied by many economists: revenue sharing arrangements (e.g., El-Hodiri & Quirk, 1971; Fort & Quirk, 1995; Vrooman, 1995; Marburger, 1997; Késenne, 2000a, 2005; Szymanski & Késenne, 2004; Dietl et al., 2011) and salary cap restrictions (e.g., Fort & Quirk, 1995; Vrooman, 1995; Késenne, 2000b; Dietl et al., 2011).

As is well known, whether or not revenue sharing improves competitive balance depends on the choice of a conjectural variation dx_j/dx_i (i ≠ j) (i.e. the rate of change in team j's choice variable anticipated by team i in response to its own change). On the one hand, it was shown that under the Walrasian conjectures $dx_j/dx_i = -1$ the introduction of revenue sharing has no impact on competitive balance in the league, i.e. the so-called "invariance proposition" (El-Hodiri & Quirk, 1971; Fort & Quirk, 1995; Vrooman, 1995; Dietl et al., 2011). On the other hand, it was shown that under the Nash conjectures $dx_j/dx_i = 0$ the invariance proposition does not hold and the introduction of revenue sharing worsens competitive balance in the league (Szymanski & Késenne, 2004; Késenne, 2005).

For a salary cap that is exogenously given, Késenne (2000b) showed that if the salary cap is binding only for a large-market team, then the introduction of the salary cap improves competitive balance. The reason is that the exogenously given salary cap reduces only the demand for talent of a large-market team. Moreover, for a salary cap and a salary floor that are exogenously given, Dietl et al. (2011) showed that if the salary cap is not binding for a large-market team and the salary floor is binding only for a small-market team, then the introduction of the salary cap and floor improves competitive balance. The reason is that the exogenously given salary floor increases only the demand for talent of a small-market team.

The purpose of this paper is to study the following fundamental problems, which have not been addressed: (i) whether or not the improvement of competitive balance leads to the maximization of the total league profits, and (ii) whether or not the optimal distribution of talent, which maximizes the total league profits, can be achieved through a process of player trading among teams under the regulatory systems mentioned above when the initial distribution of talent is different from it.

This paper is organized as follows. Section 2 discusses an N-team model where each team is a profit maximizer, each team's revenues depend only on relative team qualities, and talent supply is fixed. A salary cap studied in this paper is determined endogenously. Section 3 examines the first-order conditions for the total league profit maximization and shows a negative result about (i). Section 4 introduces incentive constraints for player trading and the concept of stability of a talent distribution. Section 5 shows a negative result about (ii).

2. The Model

2.1 Revenue Function

We consider a $\mathrm{N} \geq 3$-team league that runs a regular season every period of time t $(\mathrm{t} = 0,1, \ldots)$. In this league each team behaves as a profit maximizer, and per-period profits of each team consist of total revenue received less player salaries. The total revenue generated by team i $(\mathrm{i} = 1, \ldots, \mathrm{N})$ in period t is given by $R^i(w_i(x_t); m_i)$. Here $w_i(x_t)$ is the season win percentage of team i in period t satisfying the adding-up constraint $\sum w_k(x_t) = 1$ for all t, $x_t = (x_{1t}, \ldots, x_{Nt})$ is a distribution of playing talent in the league satisfying a constraint of fixed-supply of talent $\sum x_{kt} = X$ for all t, and m_i is a market size (or drawing potential) of team i. For simplicity, the market sizes of the teams are assumed to be constant over time and $m_1 < \cdots < m_N$. The revenue function is increasing and concave in w_i: $\partial R^i/\partial w_i > 0$ and $\partial^2 R^i/\partial w_i^2 < 0$. In addition, it is assumed that the market size has a positive effect on the marginal revenue of winning, $\partial^2 R^i/\partial w_i \partial m_i > 0$. In other words, the revenue function has strictly increasing differences in (w_i, m_i) (see Topkis (1998) and Vives (1999)).

2.2 Contest Success Function

We assume that the win percentage of a team has the following properties:

(W-1) $\partial w_i(x)/\partial x_i = -\sum_{k \neq i} \partial w_k(x)/\partial x_i$.

(W-2) $\partial w_i(x)/\partial x_i > 0$ and $\partial w_i(x)/\partial x_j < 0$, and $w_i(x) = w_j(x)$ for $i \neq j$ if $x_i = x_j$. Therefore, $w_i(x) < w_j(x)$ for $i \neq j$ if $x_i < x_j$.

(W-3) $\partial^2 w_i(x)/\partial x_i^2 < 0$, and $\partial w_i(x)/\partial x_i = \partial w_j(x)/\partial x_j$ for $i \neq j$ if $x_i = x_j$. Therefore, $\partial w_i(x)/\partial x_i > \partial w_j(x)/\partial x_j$ for $i \neq j$ if $x_i < x_j$.

(W-4) $\partial w_i(x)/\partial x_j = \partial w_i(x)/\partial x_k$ for all $i \neq j \neq k$.

(W-5) $\partial^2 w_i(x)/\partial x_i \partial x_j = \partial^2 w_i(x)/\partial x_i \partial x_k < 0$ and $\partial^2 w_i(x)/\partial x_j^2 =$

$\partial^2 w_i(x)/\partial x_j \partial x_k = \partial^2 w_i(x)/\partial x_k^2 > 0$ for all $i \neq j \neq k$.

(W-6) $w_i(\lambda x) = w_i(x)$ for all i, $\lambda > 0$, and $\lambda x = (\lambda x_1, \ldots, \lambda x_N)$.

(W-1) is the zero-sum property of contests. (W-2) is monotonicity and anonymity of the win percentage. (W-3) is concavity and anonymity of the win percentage. (W-4) represents a symmetric property of contests, which means that the effect of a talent increase in a rival $j \neq i$ on a team i's win percentage depends only on the talent level of team i. The equalities in (W-5) follow from (W-4). (W-6) is homogeneity of the win percentage.

It can be verified that the assumptions of (W-1), (W-2), and (W-6) are satisfied for the power form contest success function (CSF) $w_i(x) = x_i^\gamma / \sum x_k^\gamma$ with $\gamma > 0$.[1] Moreover, all the assumptions (W-1)-(W-6) are satisfied if $\gamma = 1$ and the conjectural variation is the Nash conjectures $dx_j/dx_i = 0$ for all $i \neq j$. The assumptions of (W-1), (W-2), (W-4), and (W-6) are satisfied, even if the conjectural variation is the Walrasian conjectures $\sum dx_k/dx_i = 0$ for all i. Under the Walrasian conjectures, we obtain $\partial^2 w_i(x)/\partial x_i^2 = 0$ and $\partial w_i(x)/\partial x_i = \partial w_j(x)/\partial x_j$ instead of (W-3) for all x, and $\partial^2 w_i(x)/\partial x_i \partial x_j = 0$ and $\partial^2 w_i(x)/\partial x_j^2 = 0$ instead of (W-5) for all x.

[1]As shown by Skaperdas (1996, Theorem 2), the power form is the only functional form that satisfies positivity and the adding-up constraint, monotonicity, anonymity, homogeneity, consistency, and independence from irrelevant alternatives (IIAs).

2.3 Revenue Sharing and Salary Caps

First, we introduce revenue sharing arrangements. Under a pool sharing system, each team retains a fraction $\mu\epsilon[0,1]$ of its own revenues and then receives $1/N$ of a revenue pool made up of all teams' revenues. Let $\bar{R}(x_t) = \sum R^k(w_k(x_t); m_k)/N$ be the total league revenue in period t divided by the number of teams N. Then the post-sharing revenue that team i receives in period t ($t \geq 0$) is $\mu R^i(w_i(x_t); m_i) + (1-\mu)\bar{R}(x_t)$.

Next, we introduce salary cap restriction. A salary cap sets a maximum amount of money that each team can spend on player salaries, and this maximum amount of money is fixed year by year as a percentage of the total league revenue in the previous season divided by the number of teams in the league. Let $\beta\epsilon(0,1)$ be the percentage of the total league revenue spent on player salaries. Then the total player salaries of team i in period t ($t \geq 1$) under the salary cap is $c_t x_{it} = \beta\bar{R}(x_{t-1})$.[2]

3. The Optimal Distribution of Talent

[2] Since the marginal revenue of a team is positive, the budget constraint of a team under the salary cap is satisfied with equality. Thus, the salary cap is binding for all teams.

3.1 Characterization of the Optimal Distribution of Talent

In this section, we will characterize the optimal talent distribution in the league, which maximizes the total league profits. The constrained profit maximization problem for the league is given by:

max $\sum_k R^k(w_k(x); m_k) - c \sum_k x_k$

s.t. $\sum_k x_k = X$,

where c is the reservation wage. The first-order conditions for a maximum are then:

$$\sum_k \frac{\partial R^k}{\partial x_1}(w_k(x^*); m_k) = \cdots = \sum_k \frac{\partial R^k}{\partial x_N}(w_k(x^*); m_k) = c^*, \tag{1}$$

$$\sum_k x_k^* = X. \tag{2}$$

The following result characterizes the optimal distribution x^* that satisfies (1).

Proposition 1. Assume that the win percentage of each team satisfies (W-1), (W-2), and (W-4). Assume also that the total league revenue is strictly concave. Then the marginal total league revenue of talent is equalized across all teams at a talent distribution x^*:

$$\sum_k \frac{\partial R^k}{\partial x_1}(w_k(x^*); m_k) = \cdots = \sum_k \frac{\partial R^k}{\partial x_N}(w_k(x^*); m_k),$$

if and only if the total league revenues are maximized at x^*:

$$\sum_k \frac{\partial R^k}{\partial x_i}(w_k(x^*); m_k) = 0 \text{ for all } i = 1, \ldots, N.$$

Proof. Since the "if" part is obvious, we prove the "only if" part. Differentiating the total league revenue with respect to x_i and using (W-1) yields

$$\sum_k \frac{\partial R^k}{\partial x_i} = \sum_{k \neq i} \left(\frac{\partial R^k}{\partial w_k} - \frac{\partial R^i}{\partial w_i} \right) \frac{\partial w_k}{\partial x_i}. \tag{3}$$

We now assume that the marginal total league revenue (3) is equalized across all teams at x^*. By (W-1) and (W-4), we obtain the symmetric relation for any i and j (i ≠ j):

$$\frac{\partial w_j}{\partial x_j} - \frac{\partial w_j}{\partial x_i} = -\sum_{k \neq j} \frac{\partial w_k}{\partial x_j} - \frac{\partial w_j}{\partial x_i} = -\frac{\partial w_i}{\partial x_j} - \sum_{k \neq i} \frac{\partial w_k}{\partial x_i} = \frac{\partial w_i}{\partial x_i} - \frac{\partial w_i}{\partial x_j}.$$

Consequently, the equality between the marginal total league revenues reduces to

$$\sum_k \frac{\partial R^k}{\partial x_i}(w_k(x^*); m_k) - \sum_k \frac{\partial R^k}{\partial x_j}(w_k(x^*); m_k) = \left(\frac{\partial R^i}{\partial w_i} - \frac{\partial R^j}{\partial w_j} \right) \left(\frac{\partial w_i}{\partial x_i} - \frac{\partial w_i}{\partial x_j} \right) = 0.$$

By (W-2), the value in the second bracket is positive. Therefore, the marginal revenue of winning is equalized across all teams at x^*:

$$\frac{\partial R^1}{\partial w_1}(w_1(x^*); m_1) = \cdots = \frac{\partial R^N}{\partial w_N}(w_N(x^*); m_N). \tag{4}$$

It follows from (3) and (4) that the total league revenues are maximized at x^*:

$$\sum_k \frac{\partial R^k}{\partial x_1}(w_k(x^*); m_k) = \cdots = \sum_k \frac{\partial R^k}{\partial x_N}(w_k(x^*); m_k) = 0 \text{ for all } i = 1, \ldots, N. \quad \square$$

Remark. A twice continuously differentiable function of N variables $f : X \to R$ is strictly concave if the Hessian matrix $H = [f_{ij}]$ of f is negative definite for every $x \in X$, where $f_{ij} \equiv \partial^2 f(x) / \partial x_i \partial x_j$ ($i, j = 1, \ldots N$). H is negative definite if H is symmetric, and has a negative dominant diagonal, i.e., if $f_{ii} < 0$ and there is $\mathrm{v} = (v_1, \dots, v_N) \gg 0$ such that $|v_i f_{ii}| > \sum_{j \neq i} |v_j f_{ij}|$ for every $\mathrm{i} = 1, \dots \mathrm{N}$ (see Mas-Colell, Whinston & Green (1995, Sections M.C and M.D)).

For the total league revenue function, the signs of diagonal elements of H are negative at x^* because of (4), $\partial^2 R^i / \partial w_i^2 < 0$, and (W-2):

$$\sum_k \frac{\partial^2 R^k}{\partial x_i^2}(w_k(x^*); m_k) = \sum_{k \neq i} \left(\frac{\partial^2 R^k}{\partial w_k^2} \frac{\partial w_k}{\partial x_i} - \frac{\partial^2 R^i}{\partial w_i^2} \frac{\partial w_i}{\partial x_i} \right) \frac{\partial w_k}{\partial x_i} < 0.$$

However, the signs of off-diagonal elements are indeterminate:

$$\sum_k \frac{\partial^2 R^k}{\partial x_j \partial x_i}(w_k(x^*); m_k) = \left(\frac{\partial^2 R^j}{\partial w_j^2} \frac{\partial w_j}{\partial x_j} - \frac{\partial^2 R^i}{\partial w_i^2} \frac{\partial w_i}{\partial x_j} \right) \frac{\partial w_j}{\partial x_i} + \sum_{k \neq i,j} \left(\frac{\partial^2 R^k}{\partial w_k^2} \frac{\partial w_k}{\partial x_j} - \frac{\partial^2 R^i}{\partial w_i^2} \frac{\partial w_i}{\partial x_j} \right) \frac{\partial w_k}{\partial x_i},$$

where the sign of the first term is positive because of $\partial^2 R^i / \partial w_i^2 < 0$ and (W-2), but the sign of the second term is indeterminate.

Let $\mathrm{TR}(x)$ be the total league revenue, and let $TR_{ij}(x) \equiv \partial^2 TR(x) / \partial x_i \partial x_j$ $(\mathrm{i}, \mathrm{j} = 1, \dots \mathrm{N})$ be its partial derivatives. Let $H_{(ss)}$ be the $(\mathrm{N}-1) \times (\mathrm{N}-1)$ submatrix of the Hessian matrix $\mathrm{H} = [TR_{ij}]$ obtained by deleting from H its s–th row and s-th column. If the off-diagonal elements of H are positive, then $H_{(ss)}$ has a negative dominant diagonal and is therefore negative definite for every $\mathrm{s} = 1, \dots, \mathrm{N}$. By (W-6), $TR_i(x) \equiv \partial TR(x) / \partial x_i$ is homogeneous of degree -1. Thus, by Euler's formula and the first-order condition $TR_i(x^*) = 0$, we obtain $\sum_{j \neq s} x_j^* TR_{ij}(x^*) = -x_s^* TR_{is}(x^*) < 0$

for $s \neq i$.[3] Since $TR_{ii}(x^*) < 0$ (i.e., $TR_{ii} = -|TR_{ii}|$) and $TR_{ij}(x^*) > 0$ (i.e., $TR_{ij} = |TR_{ij}|$), we obtain $x_i^*|TR_{ii}| > \sum_{j \neq i,s} x_j^* |TR_{ij}|$ for every $i \neq s$. □

By (W-1) and (W-2), along with (W-4), the optimal distribution $w^* = (w_1(x^*), \ldots, w_N(x^*))$ of the win percentages is uniquely determined by the equalization of the marginal revenues of winning across teams (4) and the adding-up constraint, and then the optimal distribution $x^* = (x_1^*, \ldots, x_N^*)$ of talent is uniquely determined by w^* and the constraint of fixed-supply of talent (2) in the N-team case, as in the two-team case. (Note that Proposition 1 does not require the assumptions of (W-3) and (W-5), and thus this result holds irrespective of the types of the conjectural variations dx_j/dx_i, "Walrasian" or "Nash".)

Under the concavity and complementarity assumptions imposed on the revenue function, $\partial^2 R^i/\partial w_i^2 < 0$ and $\partial^2 R^i/\partial w_i \partial m_i > 0$, and (W-2), condition (4) implies that the win percentage and the talent level of a larger-market team are higher than those of a smaller-market team: $w_1(x^*) < \cdots < w_N(x^*)$ and $x_1^* < \cdots < x_N^*$ for $m_1 < \cdots < m_N$. This implies that there exists a trade-off between the total league profit maximization and competitive balance, or uncertainty of outcome.

[3]For homogeneous functions and Euler's formula, see, for example, Mas-Colell, Whinston, & Green (1995, Section M.B).

3.2 The Implications of Proposition for Revenue Sharing and Salary Caps

We now state the implications of Proposition 1 for revenue sharing and salary caps. To do so, we consider a situation where the league has used equal revenue sharing ($\mu = 0$) since period 0, and then introduces the salary cap in period 1. The long-run profit of team i, which is defined as the sum of per-period profits, is given by:

$$\Pi^i = \bar{R}(x_0) - c_0 x_{io} + \sum_{t\geq 1}\left(\bar{R}(x_t) - \beta\bar{R}(x_{t-1})\right).$$

After rearranging terms, it is rewritten as:

$$\Pi^i = (1-\beta)\bar{R}(x_0) - c_0 x_{io} + (1-\beta)\sum_{t\geq 1}\bar{R}(x_t).$$

The talent level of each team in each period t is independently determined so that the long-run profit is maximized. The first-order conditions for the profit maximization of the teams are then given by:

$$(1-\beta)\frac{\partial\bar{R}}{\partial x_{i0}}(x_0^*) = c_0 \text{ for } i = 1,\dots,N,$$

$$(1-\beta)\frac{\partial\bar{R}}{\partial x_{it}}(x_t^*) = 0 \text{ for } i = 1,\dots,N \text{ and } t = 1,2,\dots \tag{5}$$

$$\sum x_{kt}^* = X \text{ for all } t = 0,1,2,\dots,.$$

First, a unique equilibrium distribution that satisfies (5) is x^* for all $t \geq 0$. This implies that if the salary cap is endogenously determined, then a combination of equal revenue sharing and the salary cap cannot get rid of the competitive imbalance.

More specifically, increased revenue sharing worsens the competitive balance and the introduction of the salary cap does not improve it. This contrasts with the results of Késenne (2000b) and Dietl et al. (2011) mentioned in the Introduction.

Second, the equilibrium salary per unit of talent under equal revenue sharing and without the salary cap is $c_0 = 0$. This is because increased revenue sharing reduces the marginal revenue of talent. This result implies that equal revenue sharing and salary restrictions (cap and floor) should be used simultaneously to secure a positive amount of financial resources for player salaries.

4. Player Trade and Stability of a Talent Distribution

In the previous section, we have shown that if a combination of equal revenue sharing and the salary cap is used in period $\mathrm{t} \geq 1$, then the optimal distribution of talent x^* becomes an equilibrium in period $\mathrm{t} \geq 1$. However, we should note that the salary cap cannot be used in period 0 because there is no period before period 0, and thus the total player salaries for period 0 cannot be determined appropriately. Also, as stated in Section 3.2, equal revenue sharing cannot be used without any salary restrictions (cap and floor). In the remaining sections, we will explore whether or not x^* can be achieved in period

1 through a process of player trading among teams under the restriction of the salary cap when the initial distribution in period 0, x^0, is different from x^*.

For the purpose, we introduce incentive constraints for player trading and the concept of stability of a talent distribution. The process of player trading considered here consists of successive bilateral trades between teams. A bilateral trade is defined as follows.

Definition. Let $x^{\upsilon-1} = (x_1^{\upsilon-1}, \cdots, x_N^{\upsilon-1})$ with $\sum x_k^{\upsilon-1} = X$ be a talent distribution in the league, and let $c_i^{\upsilon-1}$ be the salary per unit of talent of each team i. A bilateral trade between teams i and j (i ≠ j) in Step υ (υ = 1, 2, ...) will take place, if and only if all of the incentive constraints (a)-(c) are satisfied for a unique equilibrium distribution $x^e = (x_1^e, \cdots, x_N^e)$ with $\sum x_k^e = X$ in the player market:[4]

(a) $c_i^{\upsilon-1} > c_j^{\upsilon-1}$, (b) $x_i^{\upsilon-1} < x_i^e$, and (c) $x_j^{\upsilon-1} > x_j^e$.

The outcome of the trade in Step υ is denoted by $x^{\upsilon} = (x_1^{\upsilon}, \cdots, x_N^{\upsilon})$ with $\sum x_k^{\upsilon} = X$.

The incentive constraint (a) for players implies that players of team j have an

[4]The use of x_i^e as a target level of every team i can also be justified by the following result: if a supermodular game has a unique equilibrium, then it is dominance solvable and the unique equilibrium survives IESDS (iterated elimination of strictly dominated strategy) (see Milgrom and Roberts 1990, p.1266). For complementarity properties of the revenue function, see Appendix A2.

incentive to transfer to team i because the salary is higher in team i than in team j. The incentive constraint (b) for team i implies that team i has an incentive to acquire new players, $w_i(x^{\upsilon-1}) < w_i(x^e)$ and $\partial\pi^i/\partial x_i > 0$, because an increase in revenues brought about by an increase in the win percentage is greater than an increase in salary payments. The incentive constraint (c) for team j implies that team j has an incentive to release its players, $w_j(x^{\upsilon-1}) > w_j(x^e)$ and $\partial\pi^j/\partial x_j < 0$, because a decrease in salary payments is greater than a decrease in revenues.

A talent distribution $x^{\upsilon-1}$ is called unstable if there is at least one pair i and j that satisfies all of the constraints (a)-(c), and is stable otherwise. By definition, the equilibrium distribution x^e is stable. Since x^* is a unique equilibrium distribution under equal revenue sharing and the salary cap, it is stable under these arrangements. However, we should note that the concept of stability does not mean that a stable distribution can necessarily be achieved through the process of player trading, but rather means that once a stable distribution is achieved it is maintained afterward.

If the salary cap restriction is considered, whether or not x^* can be achieved depends on the position of the initial distribution x^0.[5] For example, we can show that if

[5]Without a salary cap, x^e is a unique stable distribution and can be achieved through the process of player trading. This is because (b) and (c) imply $\partial R^i/\partial x_i > c^e > \partial R^j/\partial x_j$ at $x^{\upsilon-1}$ for any i and j, and thus (a) can be satisfied for some $c_i > c^e > c_j$. Thus, any distribution other than x^e is unstable.

(A) $x_i^0 < x_i^*$ and $x_j^0 > x_j^*$ for all $i < J \leq j$,

then x^* can be achieved through the process of player trading. Condition (A) implies that a small-market team ($i < J$) has an excess demand for talent and a large-market team ($j \geq J$) has an excess supply of talent. As shown in Section 3.1, x^* satisfies $x_1^* < \cdots < x_N^*$ for $m_1 < \cdots < m_N$, and thus (A) also implies that a smaller-market team hires less talent than a larger-market team in period 0, i.e., $x_i^0 < x_j^0$ for all $i < J \leq j$.

We present two preliminary results. First, since the total salary payments of each team for period 1 are restricted to $c_i^{\upsilon-1}x_i^{\upsilon-1} = \beta\bar{R}(x^0)$, (a) in Definition is equivalent to $x_i^{\upsilon-1} < x_j^{\upsilon-1}$, meaning that team i can afford to acquire players of team j if and only if team i hires less talent than team j. Second, the outcome of a trade between teams i and j in Step υ is either (a') $x_i^\upsilon = x_j^\upsilon = \left(x_i^{\upsilon-1} + x_j^{\upsilon-1}\right)/2$ (i.e., $c_i^\upsilon = c_j^\upsilon$), (b') $x_i^\upsilon = x_i^*$, or (c') $x_j^\upsilon = x_j^*$. In case (A), we obtain $x_i^{\upsilon-1} < x_i^* < \left(x_i^{\upsilon-1} + x_j^{\upsilon-1}\right)/2$ if $x_i^{\upsilon-1} + x_j^{\upsilon-1} > x_i^* + x_j^*$, and $\left(x_i^{\upsilon-1} + x_j^{\upsilon-1}\right)/2 < x_j^* < x_j^{\upsilon-1}$ if $x_i^{\upsilon-1} + x_j^{\upsilon-1} < x_i^* + x_j^*$. This implies that one of the two teams that make a trade can achieve the optimal talent level.

The result mentioned above is verified as follows: **Step 1.** Since (a) $x_i^0 < x_j^0$ (i.e., $c_i^0 > c_j^0$), (b) $x_i^0 < x_i^*$, and (c) $x_j^0 > x_j^*$, a trade takes place between teams i and j ($i < J \leq j$). If $x_i^0 + x_j^0 > x_i^* + x_j^*$, the outcome of the trade is (b') $x_i^1 = x_i^*$ and (c) $x_j^1 = x_j^0 - (x_i^* - x_i^0) > x_j^*$ (i.e., $c_i^1 > c_j^1$). **Step 2.** Since team j does not reduce its talent to the optimal level yet, a new trade subsequently takes place between team j and some team $h\,(< J)$ with (a) $x_h^1 < x_j^1$ (i.e., $c_h^1 > c_j^1$) and (b) $x_h^1 < x_h^*$. If $x_h^1 + x_j^1 < x_h^* + x_j^*$, then the outcome of the trade is (b) $x_h^2 = x_h^1 + \left(x_j^1 - x_j^*\right) < x_h^*$ and (c') $x_j^2 = x_j^*$ (i.e., $c_h^2 > c_j^2$), and so on. **Step N-1.** The process of successive bilateral trades terminates when a stable distribution is reached. By the constraint of fixed-supply of talent, $\sum_{i<J}(x_i^* - x_i^0) = \sum_{j\geq J}\left(x_j^0 - x_j^*\right)$, x^* is achieved on termination.

5. A No-trade Result and the Role of a Salary Cap

In this section, we will show that when the league used a pool sharing in period 0, and then introduces a combination of equal revenue sharing and a salary cap in period 1, the optimal distribution x^* cannot be achieved through the process of off-season player trading among teams under the newly introduced salary cap restriction.

For the purpose, we consider a situation where the league used the pool sharing with a share parameter $\mu \in (0,1)$ in period 0, and then introduces a combination of equal revenue sharing and the salary cap with a share parameter $\beta \in (0,1-\mu)$ in period 1. After rearranging terms, the long-run profit of team i is written as:

$\Pi^i = \hat{R}^i(x_0) - c_\mu x_{i0} + (1-\beta)\sum_{t\geq 1} \bar{R}(x_t),$

where $\hat{R}^i(x_0) = \mu R^i(w_i(x_0); m_i) + (1 - \mu - \beta)\bar{R}(x_0)$ is the post-sharing revenue of team i in period 0 and c_μ is the salary per unit of talent under the pool sharing. The first-order conditions for the profit maximization of the teams are then given by:

$$\frac{\partial \hat{R}^i}{\partial x_{i0}}(x_0^*) = c_\mu \text{ for } \mathrm{i} = 1, \dots, \mathrm{N}, \tag{6}$$

$$\frac{\partial \bar{R}}{\partial x_{it}}(x_t^*) = 0 \text{ for } \mathrm{i} = 1, \dots, \mathrm{N} \text{ and } \mathrm{t} = 1, 2, \dots, \tag{7}$$

$$\sum x_{kt}^* = X \text{ for all } \mathrm{t} = 0, 1, 2, \dots,. \tag{8}$$

By Proposition 1, (6), and (7), a unique equilibrium distribution is $x_0^* \neq x^*$ for $\mathrm{t} = 0$ and $x_t^* = x^*$ for $\mathrm{t} \geq 1$. For notational simplicity, we write $x^0 \equiv x_0^*$.

The following result shows the relation between x^0 and x^*.

Proposition 2. Assume that the win percentage of each team satisfies (W-1)-(W-6). Assume also that the post-sharing revenue satisfies (I) $\partial^2 \hat{R}^i(x^0)/\partial x_i^2 < 0$ and (II) $\partial^2 \hat{R}^i(x^0)/\partial x_i \partial x_j > 0$ at x^0 for $\mathrm{i} \neq \mathrm{j}$ and $\mathrm{i}, \mathrm{j} = 1, \dots, \mathrm{N}$. Then there is a threshold $1 < \mathrm{J} < \mathrm{N}$ such that

(B) (i) $x_i^0 < x_j^0$ and (ii) $x_i^0 > x_i^*$ and $x_j^0 < x_j^*$ for all $\mathrm{i} < \mathrm{J} \leq \mathrm{j}$.

The proof of Proposition 2 is given in the Appendix A1. The implications of Proposition 2 are stated as follows. First, (i) and (ii) of condition (B) imply that x^0 is stable in period 1 and is maintained afterward, even if it is no longer an equilibrium distribution after the regime shift. The reason for the stability is that under the salary cap (i) is equivalent to $c_i^0 > c_j^0$ for all $i < J \leq j$, and the salary floor prevents small-market teams ($i < J$) with an excess supply of talent ($x_i^0 > x_i^*$) from lowering c_i, and the salary cap prevents large-market teams ($j \geq J$) with an excess demand for talent ($x_j^0 < x_j^*$) from raising c_j, and thus no player trading takes place between a small-market team and a large-market team.

Second, since (ii) of condition (B) states that increased revenue sharing worsen competitive balance in equilibrium, the no-trade result implies that the role of the salary cap is not to increase the total league profits, but to maintain (not improve) competitive balance in the league.

Third, Proposition 2 implies that two-team models can be justified as a first approximation of the N-team model in studying the impact of revenue sharing. In this proposition, we can divide the teams in the N-team league into two groups: a group of small-market teams ($i < J$) and a group of large-market teams ($j \geq J$). Since (ii) of condition (B) implies that all small-market (large-market) teams respond to increased revenue sharing in the same direction, a small-market team and a large-market team in a two-team model can be regarded as a representative team of each group in the N-team model, respectively.

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Appendix A1. Proof of Proposition 2

First, we show under (W-1)-(W-4) that (i) $x^0 = (x_1^0, \cdots, x_N^0)$ satisfies $x_1^0 < \cdots < x_N^0$ for $m_1 < \cdots < m_N$. We assume, by way of contradiction, that there exist i and j such that $x_i^0 \geq x_j^0$ and $m_i < m_j$. As in the proof of Proposition 1, by (W-1) and (W-4), the first-order conditions (6) for teams i and j reduce to

$$\frac{\partial R^i}{\partial w_i}\left[\mu \frac{\partial w_i}{\partial x_i} + \Phi(\mu, \beta)\left(\frac{\partial w_i}{\partial x_i} - \frac{\partial w_i}{\partial x_j}\right)\right] = \frac{\partial R^j}{\partial w_j}\left[\mu \frac{\partial w_j}{\partial x_j} + \Phi(\mu, \beta)\left(\frac{\partial w_i}{\partial x_i} - \frac{\partial w_i}{\partial x_j}\right)\right], \tag{9}$$

where $\Phi(\mu, \beta) = (1 - \mu - \beta)/N > 0$. By (W-3), $x_i^0 \geq x_j^0$ implies $\partial w_i/\partial x_i \leq \partial w_j/\partial x_j$. Thus, by (9), $\partial R^i/\partial w_i \geq \partial R^j/\partial w_j$ must hold at x^0 for any $0 < \mu < 1$. On the other hand, by (W-2) and $\partial^2 R^i/\partial w_i^2 < 0$ and $\partial^2 R^i/\partial w_i \partial m_i > 0$, $x_i^0 \geq x_j^0$ and $m_i < m_j$ implies that $\partial R^i/\partial w_i < \partial R^j/\partial w_j$ at x^0. Contradiction. Therefore, the desired result is obtained. □

Next, we show under (W-1)-(W-6) that (ii) there is a thresholds $1 < J < N$ such that $x_i^0 > x_i^*$ for $i < J$ and $x_i^0 < x_i^*$ for $i \geq J$. To do so, we investigate the shifts of the teams' demand curves for talent caused by increased revenue sharing. Differentiating $\partial\hat{R}^i/\partial x_i$ given in (6) with respect to μ and then using (7) to evaluate the result at x^* gives us:

$$\frac{\partial}{\partial\mu}\frac{\partial\hat{R}^i}{\partial x_i}(x^*) = \frac{\partial R^i}{\partial w_i}(w_i(x^*); m_i)\cdot\frac{\partial w_i}{\partial x_i}(x^*) > 0. \tag{10}$$

As shown in Section 3.1, by (W-1), (W-2), and (W-4), $\partial R^1/\partial w_1 = \cdots = \partial R^N/\partial w_N$ at x^* (see (4)) and $x_1^* < \cdots < x_N^*$ for $m_1 < \cdots < m_N$. In addition, by (W-3), we obtain $\partial w_i(x^*)/\partial x_i > \partial w_j(x^*)/\partial x_j$ for $x_i^* < x_j^*$. Therefore, it follows from (10) that the size of the downward shift of the demand curve evaluated at x^* when the share parameter is reduced by μ is greater for a smaller-market team than for a larger-market team:

$$\mu\frac{\partial}{\partial\mu}\frac{\partial\hat{R}^1}{\partial x_1}(x^*) > \cdots > \mu\frac{\partial}{\partial\mu}\frac{\partial\hat{R}^N}{\partial x_N}(x^*) \text{ for } m_1 < \cdots < m_N \text{ and } 0 < \mu < 1, \tag{11}$$

where $\mu\,\partial\big(\partial\hat{R}^i(x^*)/\partial x_i\big)/\partial\mu = \partial\hat{R}^i(x^*)/\partial x_i$.

Define a system of differential equations $\dot{x}_i = Z^i(x)$ $(\mathrm{i} = 1, \dots, \mathrm{N})$ over the simplex $\sum x_i = X$, where $Z^i(x) = \partial\hat{R}^i(x)/\partial x_i - c_\mu$. By (6), x^0 is a unique stationary point of the system. By (8), we have $\sum\dot{x}_i = \sum Z^i(x) = 0$. It follows from this and (11) that there exists a threshold $1 < \mathrm{J} < \mathrm{N}$ such that $Z^i(x^*) > 0$ for all $\mathrm{i} < \mathrm{J}$ and $Z^i(x^*) < 0$ for all $\mathrm{i} \geq \mathrm{J}$ (Since $x^0 \neq x^*$, by (6) and (7), there is no i such that $Z^i(x^*) = 0$). Thus, if x^0 is locally and asymptotically stable under the differential equation system above, then the talent level increases $x_i^* < x_i^0$ for $\mathrm{i} < \mathrm{J}$ and decreases

$x_i^* > x_i^0$ for $\mathrm{i} \geq \mathrm{J}$ along the paths from x^* to x^0 for sufficiently small $\mu > 0$.

We show that if conditions (I) and (II) are satisfied, then x^0 is locally and asymptotically stable. By (W-6), Z^i is homogeneous of degree -1. Thus, by Euler's formula and $Z^i(x^0) = 0$, we obtain $\sum_{j \neq N} x_j^0 Z_j^i(x^0) = -x_N^0 Z_N^i(x^0) < 0$, where $Z_j^i \equiv \partial Z^i / \partial x_j$. Since conditions (I) and (II) respectively imply that $Z_i^i(x^0) < 0$ (i.e., $Z_i^i = -|Z_i^i|$) and $Z_j^i(x^0) > 0$ (i.e., $Z_j^i = |Z_j^i|$) at x^0 for $\mathrm{i} \neq \mathrm{j}$ and $\mathrm{i}, \mathrm{j} = 1, \dots, \mathrm{N}$, the reduced Jacobian matrix $[Z_j^i]$ of $Z^i(x)$ $(\mathrm{i}, \mathrm{j} = 1, \dots, \mathrm{N} - 1)$ has a negative dominant diagonal, i.e., $x_i^0 |Z_i^i| > \sum_{j \neq i, N} x_j^0 |Z_j^i|$, and thus all its characteristic roots have negative real parts. Therefore, the desired result is obtained. □

We will examine conditions (I) and (II) using (W-2)-(W-5) in the next section.

The concavity of the CSF (W-3) is the key assumption for (11) in studying the impact of revenue sharing. Three remarks on (W-3) are in order.

Remark 1. Whether or not the invariance proposition holds with respect to revenue sharing simply depends on the shape of the CSF. (Note that this is also true in the case without a salary cap.) As shown in Section 2.2, if we assume the Walrasian conjectures $\sum dx_k / dx_i = 0$ for all i, then we have $\partial w_i(x) / \partial x_i = \partial w_j(x) / \partial x_j$ for any x and $\mathrm{i} \neq$

j instead of (W-3). By (4) and (10), $\mu\,\partial\big(\partial\hat{R}^i(x^*)/\partial x_i\big)/\partial\mu = \mu\,\partial\big(\partial\hat{R}^j(x^*)/\partial x_j\big)/\partial\mu$ holds for any $0 < \mu < 1$ and $i \neq j$ instead of (11), meaning that the size of the shift of the demand curve for talent evaluated at x^* is equal for all teams. Thus, revenue sharing has no impact on the distribution of talent in the league. □

Remark 2. Szymanski (2013, p.324) used a concave CSF to reject the invariance proposition. Madden (2015, p.546) also used the concavity of the CSF implicitly to prove his Propositions 1 and 2. □

Remark 3. Késenne (2005) showed that the sizes of the shifts of the demand curves at an equilibrium distribution $x^e = (x_1^e, \ldots, x_N^e)$ without revenue sharing ($\mu = 1$) are

$\frac{\partial}{\partial\mu}\frac{\partial\hat{R}^i}{\partial x_i}(x^e) > \frac{\partial}{\partial\mu}\frac{\partial\hat{R}^j}{\partial x_j}(x^e)$ if $x_i^e < x_j^e$,

where $\frac{\partial}{\partial\mu}\frac{\partial\hat{R}^i}{\partial x_i} = \frac{\partial R^i}{\partial x_i} - \frac{1}{N}\sum_k \frac{\partial R^k}{\partial x_i} = c^e\left[\frac{N-1}{N} - \frac{1}{N}\sum_{k\neq i}\frac{\partial w_k}{\partial x_i}\Big/\frac{\partial w_k}{\partial x_k}\right] > 0$, $c^e > 0$ is the salary per unit of talent without revenue sharing, and $w_i = x_i/\sum x_k$ is the CSF.

We can prove this result using (W-3). By (W-3), we obtain

$$\frac{\partial w_i}{\partial x_i}(x^e) > \frac{\partial w_j}{\partial x_j}(x^e) \text{ if } x_i^e < x_j^e. \tag{12}$$

By the symmetric relation given in the proof of Proposition 1 $\partial w_j/\partial x_j - \partial w_j/\partial x_i = \partial w_i/\partial x_i - \partial w_i/\partial x_j$ and (W-2), (12) implies $\frac{\partial w_j}{\partial x_i}\Big/\frac{\partial w_j}{\partial x_j} < \frac{\partial w_i}{\partial x_j}\Big/\frac{\partial w_i}{\partial x_i}$ at x^e. Therefore, by

(W-4), we obtain

$$\sum_k \frac{\partial R^k}{\partial x_i}(w_k(x^e); m_k) - \sum_k \frac{\partial R^k}{\partial x_j}(w_k(x^e); m_k) = c^e \left[\frac{\partial w_j}{\partial x_i}\Big/\frac{\partial w_j}{\partial x_j} - \frac{\partial w_i}{\partial x_j}\Big/\frac{\partial w_i}{\partial x_i}\right] < 0.$$

The desired result follows from this.

We can also prove the result as follows. By the first-order conditions without revenue sharing, $\partial R^i/\partial w_i \cdot \partial w_i(x^e)/\partial x_i = \partial R^j/\partial w_j \cdot \partial w_j(x^e)/\partial x_j$ at x^e, (12) implies $\partial R^i/\partial w_i < \partial R^j/\partial w_j$ at x^e. By an argument exactly analogous to that used in the proof of Proposition 1, we obtain

$$\sum_k \frac{\partial R^k}{\partial x_i}(w_k(x^e); m_k) - \sum_k \frac{\partial R^k}{\partial x_j}(w_k(x^e); m_k) = \left(\frac{\partial R^i}{\partial w_i} - \frac{\partial R^j}{\partial w_j}\right)\left(\frac{\partial w_i}{\partial x_i} - \frac{\partial w_i}{\partial x_j}\right) < 0.$$

The desired result follows from this.

By an argument exactly analogous to that used in the proof (i) of Proposition 2, we obtain $x_1^e < \cdots < x_N^e$ for $m_1 < \cdots < m_N$. Consequently, we obtain

$$(1-\mu)\frac{\partial}{\partial \mu}\frac{\partial \hat{R}^1}{\partial x_1}(x^e) > \cdots > (1-\mu)\frac{\partial}{\partial \mu}\frac{\partial \hat{R}^N}{\partial x_N}(x^e) \text{ for } m_1 < \cdots < m_N \text{ and } 0 < \mu < 1,$$

where $(1-\mu)\,\partial\big(\partial \hat{R}^i(x^e)/\partial x_i\big)/\partial \mu = c^e - \partial \hat{R}^i(x^e)/\partial x_i$.

From the above argument, if (W-3) is satisfied, then a talent increase in a smaller-market team has a greater negative impact on the total league revenue than a talent increase in a larger-market team does because of $\partial w_j/\partial x_i < \partial w_i/\partial x_j < 0$ and $\partial R^i/\partial w_i < \partial R^j/\partial w_j$ for $m_i < m_j$, and thus the size of the downward shift of the demand curve evaluated at x^e is greater for a smaller-market team than for a larger-market team. (Contrary to Késenne's (2007, p.117) conjecture, a smaller- (larger-) market team is a team with less (more) talent in this N-team model). □

Appendix A2. Conditions (I) and (II) in Proposition 2: Monotonicity of Marginal Revenues

Here we examine conditions (I) and (II). Changes in the marginal post-sharing revenue are given by:

$$Z_i^i = \frac{\partial^2 \hat{R}^i}{\partial x_i^2} = [\mu + \Phi(\mu,\beta)]\frac{\partial^2 R^i}{\partial x_i^2} + \Phi(\mu,\beta)\frac{\partial^2 R^j}{\partial x_i^2} + \Phi(\mu,\beta)\sum_{k\neq i,j}\frac{\partial^2 R^k}{\partial x_i^2}, \qquad (13)$$

$$Z_j^i = \frac{\partial^2 \hat{R}^i}{\partial x_i \partial x_j} = [\mu + \Phi(\mu,\beta)]\frac{\partial^2 R^i}{\partial x_i \partial x_j} + \Phi(\mu,\beta)\frac{\partial^2 R^j}{\partial x_i \partial x_j} + \Phi(\mu,\beta)\sum_{k\neq i,j}\frac{\partial^2 R^k}{\partial x_i \partial x_j}, \qquad (14)$$

where $\Phi(\mu,\beta) = (1-\mu-\beta)/N > 0$. Each term in (13) is written as:

$$\frac{\partial^2 R^l}{\partial x_i^2} = \frac{\partial^2 R^l}{\partial w_l^2}\left(\frac{\partial w_l}{\partial x_i}\right)^2 + \frac{\partial R^l}{\partial w_l}\frac{\partial^2 w_l}{\partial x_i^2} \text{ for } \mathrm{l} = 1, \ldots, \mathrm{N}. \qquad (15)$$

Each term in (14) for $\mathrm{i} \neq \mathrm{j}$ is written as:

$$\frac{\partial^2 R^l}{\partial x_i \partial x_j} = \frac{\partial^2 R^l}{\partial w_l^2}\frac{\partial w_l}{\partial x_i}\frac{\partial w_l}{\partial x_j} + \frac{\partial R^l}{\partial w_l}\frac{\partial^2 w_l}{\partial x_i \partial x_j} \text{ for } \mathrm{l} = 1, \ldots, \mathrm{N}. \qquad (16)$$

By (W-4) and (W-5), the first and second terms on the right-hand side of (15) respectively equals those of (16) if $\mathrm{l} = \mathrm{k} \neq \mathrm{i}, \mathrm{j}$.

First, we examine condition (I). By the concavity of the revenue function, the first term in (15) is negative for all $\mathrm{l} = 1, \ldots, \mathrm{N}$. By (W-3) and (W-5), the second term in

(15) is negative if $l = i$, and is positive if $l \neq i$. Therefore, the value of (15) is negative, if the first term is greater than the second term for $l \neq i$.

Next, we move to condition (II). By (W-2) and (W-4), the first term in (16) is positive if $l = i$ or $l = j$, and is negative if $l \neq i, j$. By (W-5), the second term in (16) is negative if $l = i$ or $l = j$, and is positive if $l \neq i, j$. Therefore, the value of (16) is positive, if the first term is greater than the second term for $l = i$ or $l = j$, and the second term is greater than the first term for $l \neq i, j$.

For example, for the constant-elasticity revenue function $R^i = m_i w_i^{\varepsilon}$ with $0 < \varepsilon < 1$ and the power form CSF $w_i = x_i / \sum x_l$, (15) are

$$\frac{\partial^2 R^j}{\partial x_i^2} = \frac{\varepsilon(\varepsilon+1) m_j x_j^{\varepsilon}}{(\sum x_l)^{\varepsilon+2}} > 0 \text{ and } \frac{\partial^2 R^k}{\partial x_i^2} = \frac{\partial^2 R^k}{\partial x_i \partial x_j} = \frac{\varepsilon(\varepsilon+1) m_k x_k^{\varepsilon}}{(\sum x_l)^{\varepsilon+2}} > 0.$$

Also, (16) are

$$\frac{\partial^2 R^i}{\partial x_i \partial x_j} = \frac{\varepsilon m_i x_i^{\varepsilon-1} [x_i - \varepsilon \sum_{l \neq i} x_l]}{(\sum x_l)^{\varepsilon+2}},$$

which are positive only if $0 < \varepsilon < 1/(N-1)$.

For the quadratic revenue function $R^i = m_i w_i - \alpha w_i^2$ with $\alpha > 0$ and the power form CSF $w_i = x_i / \sum x_l$, (15) are

$$\frac{\partial^2 R^j}{\partial x_i^2} = \frac{2x_j [m_j \sum x_l - 3\alpha x_j]}{(\sum x_l)^4} \text{ and } \frac{\partial^2 R^k}{\partial x_i^2} = \frac{\partial^2 R^k}{\partial x_i \partial x_j} = \frac{2x_k [m_k \sum x_l - 3\alpha x_k]}{(\sum x_l)^4},$$

which are negative only if $\alpha > \sum m_l / 3$. Also, (16) are

$$\frac{\partial^2 R^i}{\partial x_i \partial x_j} = \frac{2\alpha x_i [2\sum x_l - 3x_i] - m_i \sum x_l [\sum x_l - 2x_i]}{(\sum x_l)^4},$$

which are positive only if $\alpha > \sum m_i[(1 - 2w_i)/\sum 2w_l(2 - 3w_l)]$.

In both examples, conditions (I) and (II) are satisfied only if the marginal revenue of winning declines rapidly as the win percentage increases.