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Abstract: Weber (*The Manchester School*, Vol. 69 (2001), No. 6, pp. 616-622) has asserted that if we treat the production function of Epstein and Spiegel (*The Manchester School*, Vol. 68 (2000), No. 5, pp. 503-515) as a utility function, then the inferior good exhibits the Giffen behavior at certain positive prices and income levels. This paper shows that it is not correct. Furthermore, we develop a geometrical method in general two-goods settings to draw the rough shape of a price-consumption curve, and to calculate the ranges of prices, income levels, and income shares at which the Giffen behavior appears, and to characterize utility functions that generate the Giffen behavior.

Keywords: Giffen behavior; Marginal rate of substitution; Geometric condition; Price-consumption curve; Income-consumption curve

JEL classification: D11

1 INTRODUCTION

In a 2001 paper in this journal, Weber (2001) asserted that if we treat the E-S (Epstein and Spiegel (2000)) production function as a utility function, then the inferior good exhibits the Giffen behavior at certain prices and income levels. However, his argument is not correct in the following two points. First, his necessary and sufficient condition for the Giffen behavior (Eq.(12) in p.621) is not correct. The correct condition can be obtained not only by the implicit function theorem, as used in Weber (2001), but also by the necessary and sufficient geometric condition given by Vandermeulen (1972). Second, the inferior good is not Giffen at positive prices and income levels. It is shown that the necessary and sufficient condition for the Giffen behavior in the case of the E-S utility function is satisfied only in a region of a commodity space where the marginal utility of an inferior good (good 1) is negative. In that region upward-sloping budget lines are tangent to upward-sloping indifference curves. Reexamination of the E-S utility function reveals the importance of checking how the marginal rate of substitution (MRS) changes not only along each indifference curve (the second-order conditions for a maximum) but also over an entire commodity space before solving utility-maximization problems and doing comparative statics analysis.

In order to avoid the above mentioned errors, this paper develops in general two-good settings a geometrical method using the geometric condition of Vandermeulen (1972), which focuses on how the MRS changes along each vertical line, as well as the second-order conditions. Using this method, we can identify in advance not only a region in a commodity space where indifference curves are upward-sloping or concave to the origin but also a region where indifference curves are convex to the origin and a price-consumption curve (PCC) becomes backward-sloping. Furthermore, we can calculate using this method the ranges of relative prices and income levels at which the Giffen behavior appears and the range of income shares for the Giffen good. The great advantage of using our geometrical method is that we can know how the Giffen behavior appears or disappear as the price of a good or income level change, even if the first-order conditions of the utility-maximization problem cannot be solved for demand functions explicitly.

This paper is organized as follows. In Section 2 we prove the above mentioned two results by using the implicit function theorem. In Section 3 we develop a geometrical method and then prove the latter result by using the method. Subsequently, we compare the E-S utility function with that of Vandermeulen (1972) to point out the causes of the latter negative result, and then present a class of new Giffen utility functions. In Section 4 we apply our geometrical method to the well-known additive-separable functions, those of Silberberg and Walker (1984) and Spiegel (1994), to present some new results. As we will see later, we can analyze the E-S function, the new Giffen functions, and additive-separable functions in similar ways. Another implication of the results obtained in Sections 3 and 4 is that under certain parametric assumptions representing asymmetry between goods a combination of standard functional forms, e.g., the Modified Bergson (or the HARA (hyperbolic absolute risk aversion)) class, CARA (constant absolute risk aversion), and CRRA (constant relative risk aversion), can generate the Giffen behavior.

2 THE RESULTS

We now consider a constrained utility-maximization problem for a consumer, in which the consumer purchases goods 1 and 2 at exogenously given positive prices $p > 0$ and $q > 0$ respectively, and chooses the quantities of good 1 $x \in X$ and good 2 $y \in Y$ demanded so as to maximize utility. We assume that the utility function has the E-S (Epstein and Spiegel, 2000) form. Then the utility-maximization problem becomes:

$$\max_{\{x,y\}} U = \delta \frac{(x+y)^{\gamma}}{\gamma} - \beta \frac{x^{\alpha}}{\alpha}$$

s.t. $\quad px + qy = M,$

where $M > 0$ is a consumer's positive constant income. Assume that parameters of the utility function satisfy $\delta > 0$, $0 < \gamma < 1$, $\beta > 0$, and $\alpha > 1$, as in Epstein and Spiegel (2000) and Weber (2001).

The partial derivatives thus are:

$$U_x = \delta(x+y)^{\gamma-1} - \beta x^{\alpha-1};$$

$$U_{xx} = (\gamma-1)\delta(x+y)^{\gamma-2} - (\alpha-1)\beta x^{\alpha-2} < 0;$$

$$U_{xy} = (\gamma-1)\delta(x+y)^{\gamma-2}; \tag{1}$$

$$U_y = \delta(x+y)^{\gamma-1} > 0; \quad U_{yy} = (\gamma-1)\delta(x+y)^{\gamma-2} < 0.$$

As shown by Epstein and Spiegel (2000), the bordered Hessian is always positive under the above parametric assumptions:

$$\varepsilon = -U_{xx}U_y^2 + 2U_{xy}U_xU_y - U_{yy}U_x^2$$

$$= (\alpha-1)\beta x^{\alpha-2}\delta^2(x+y)^{2\gamma-2} + \beta^2 x^{2\alpha-2}(1-\gamma)\delta(x+y)^{\gamma-2} > 0.$$

The first-order conditions for a maximum, which together imply the equality between the marginal rate of substitution (MRS) and the relative price, are given by:

$$1 - \frac{\beta x^{\alpha-1}}{\delta(x+y)^{\gamma-1}} = \frac{p}{q} \tag{2}$$

Since both of the prices are positive, $p > 0$ and $q > 0$, the condition:

$$\beta x^{\alpha-1} < \delta(x+y)^{\gamma-1}, \text{ or equivalently, } U_x > 0 \tag{3}$$

must be satisfied for equilibrium points located on downward-sloping budget lines.

In what follows, we prove that good 1 is an inferior good, but not a Giffen good, by using the implicit function theorem. Solving the budget constraint for y and then substituting the result into (2) yields:

$$f(x,p,q,M) \equiv \frac{p}{q} - 1 + \frac{\beta x^{\alpha-1}}{\delta[M/q+(1-p/q)x]^{\gamma-1}} = 0.$$

(Eq.(10) in Weber (2001, p.620) is not correct.) By the implicit function theorem, which is also used by Weber (2001), we obtain

$$\frac{\partial x(p,q,M)}{\partial p} = -\frac{\partial f/\partial p}{\partial f/\partial x}$$

$$= \frac{-\delta[M/q+(1-p/q)x]^{\gamma}+(1-\gamma)\beta x^{\alpha}}{q\beta x^{\alpha-1}\{[M/q+(1-p/q)x](\alpha-1)/x+(1-\gamma)(1-p/q)\}}.$$

From the parametric assumptions mentioned above, both terms in the denominator are positive. Thus, the demand curve for good 1 is upward-sloping if and only if

$$(1-\gamma)\beta x^{\alpha} > \delta[M/q + (1-p/q)x]^{\gamma} \qquad (4)$$

(Eq.(12) in Weber (2001, p.621) is not correct.) Substituting $M/q - (p/q)x = y$ into (4) again, we can rewrite the condition (4) as:

$$(1-\gamma)\beta x^{\alpha} > \delta(x+y)^{\gamma} \qquad (5)$$

This necessary and sufficient condition (5) for the Giffen behavior can also be obtained from the necessary and sufficient geometric condition, given by Vandermeulen (1972):

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right) = (1-\gamma)\frac{\beta x^{\alpha-1}}{\delta(x+y)^{\gamma}} > \frac{1}{x} = -\frac{\partial}{\partial y}\left(\frac{p}{q}\right).$$

As explained by Vandermeulen (1972, p.454), this geometric condition states that the quantity demanded of good 1 is reduced as its price is reduced if and only if indifference curves flatten out along the vertical lines more rapidly than the rotating budget line.

We now assume that a certain consumption bundle $(x,y) \in X \times Y$ satisfies (5). Since the inequality $x + y > (1-\gamma)x$ holds for any $x > 0$, $y > 0$, and $0 < \gamma < 1$, we have $\delta(x+y)^{\gamma} > (1-\gamma)x\delta(x+y)^{\gamma-1}$. Combining this inequality with (5) gives us $\beta x^{\alpha-1} > \delta(x+y)^{\gamma-1}$. It follows from this and (1) that the marginal utility of good 1 is negative, and thus an indifference curve is upward-sloping at that point (x, y). This result and (3) together imply that if a certain consumption bundle satisfies (5), then the bundle cannot be an equilibrium point for any pair of positive prices and positive income level. Therefore, the demand curve for good 1 is always downward-sloping.

However, it can be verified that good 1 is always an inferior good for $0 < \gamma < 1$ and $\alpha > 1$ since

$$\frac{\partial x(p,q,M)}{\partial M} = \frac{\gamma-1}{q\{[M/q+(1-p/q)x](\alpha-1)/x+(1-\gamma)(1-p/q)\}}<0.$$

3 A GEOMETRICAL APPROACH TO THE GIFFEN BEHAVIOR

3.1 Geometrical Method and Some General Results

The result that the E-S utility function does not generate the Giffen behavior is also verified in the following manner. Define the subset of a commodity space $X \times Y$ where the marginal utility of each good is positive, $\Lambda_{++} \equiv \{(x, y) \in X \times Y | U_x > 0 \ and \ U_y > 0\}$. We then divide the subset Λ_{++} according to the value of a change in the MRS function (a CMRS function, hereafter), $-\partial(U_x/U_y)/\partial y : \Lambda_{++} \to R$. There are at most three critical values in the range of the CMRS function,

$$0 < \frac{1}{x} < -\frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right)(x, y) \cdot \frac{U_y}{U_x}(x, y).$$

(Note that when good 2 is a superior good the MRS monotonically decreases as x increases, i.e., $\partial(U_x/U_y)/\partial x = (U_{xx}U_y - U_x U_{yx})/U_y^2 < 0$.) Accordingly, the subset Λ_{++} can be divided into at most four regions. First, (Region-C) is the set of $(x, y) \in \Lambda_{++}$ such that the value of the CMRS function is greater than the critical value on the right-hand side. In this region, indifference curves are concave to the origin and thus the optimal consumption bundle is a corner solution because an increase in the MRS when y is reduced is greater than a decrease in the MRS when x is increased, so that

$$\frac{d}{dx}\left(\frac{U_x}{U_y}\right)\bigg|_{U=const.}=\frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right)(x,y)-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x,y)\cdot\frac{U_x}{U_y}(x,y)>0;$$

Second, (Region-G) is the set of $(x,y)\in\Lambda_{++}$ such that the value of the CMRS function is between $1/x$ and the critical value on the right-hand side. In this region, indifference curves are convex to the origin and good 1 is a Giffen good, as suggested by the geometric condition of Vandermeulen (1972):[1]

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x,y)=\frac{U_xU_{yy}-U_{xy}U_y}{U_y^2}>\frac{1}{x}=-\frac{\partial}{\partial y}\left(\frac{p}{q}\right);$$

Third, (Region-I) is the set of $(x,y)\in\Lambda_{++}$ such that the value of the CMRS function is between zero and $1/x$. In this region, indifference curves are convex to the origin and good 1 is an inferior good; Lastly, (Region-S) is the set of $(x,y)\in\Lambda_{++}$ such that the value of the CMRS function is negative. In this region, the income effect on the demand for good 1 is positive and thus good 1 is a superior good since

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x,y)=\frac{U_xU_{yy}-U_{xy}U_y}{U_y^2}<0.$$

As a result of such division, we can know how the Giffen behavior appears or disappears as relative price of goods or income level change. First, by checking the range of the CMRS function $-\partial(U_x/U_y)/\partial y$ and the sign of its partial derivative with respect to y, $-\partial^2(U_x/U_y)/\partial y^2$, we can know the number of regions the subset Λ_{++} is divided into and the location of (Region-G) in the commodity space. The results for the case of $1/x<-\partial(U_x/U_y)/\partial x\cdot(U_y/U_x)$ are presented as follows.[2]

[1]This result can be verified by using the Slutsky equation derived from the first-order conditions. For the Slutsky equation, see, for example, Varian (1992, pp.123-124). The n-goods version of the geometric condition is given by Brown (2000, Theorem 14, p.347).

[2]As is verified by using the Slutsky equation, this condition is equivalent to $\partial y/\partial p<0$,

Proposition 1: (a) If a decline in the absolute value of the MRS decreases as the quantity of good 2 is increased, $-\partial^2(U_x/U_y)/\partial y^2 < 0$, then (Region-G) is north of (Region-C). Moreover, if the MRS is not zero on the upper boundary of (Region-G), then (Region-G) is south of (Region-I).

(b) If a decline in the absolute value of the MRS increases as the quantity of good 2 is increased, $-\partial^2(U_x/U_y)/\partial y^2 > 0$, then (Region-G) is north of (Region-I). Moreover, if the MRS is not zero on the upper boundary of (Region-G), then (Region-G) is south of (Region-C).

Examples of case (a) are Vandermeulen (1972), Silberberg and Walker (1984), and Spiegel (1994) (see Figure 1 of Vandermeulen (1972, p.455) and Figures 5 and 6 in Section 4). Examples of case (b) are Doi et al. (2009), Moffatt (2012), and Moffatt and Moffatt (2014) (see Figure 3b in Doi et al. (2009, p.259)). If we interpret the lower boundary of (Region-G) as the implied subsistence constraint, (Region-C) as the calorie-deprived zone, (Region-G) as the subsistence zone, and (Region-I) as the standard zone, then the results obtained for case (a) are consistent with an inverted-U pattern, which was discovered empirically by Jensen and Miller (2008), whereas the results obtained for case (b) are not.[3]

Next, by calculating the MRS for equilibrium points located on the upper boundary $y = g_u(x)$ and the lower boundary $y = g_l(x)$ of (Region-G) respectively, we obtain the

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meaning that good 2 is a gross complement to good 1. For the case of $-\partial(U_x/U_y)/\partial x \cdot (U_y/U_x) \leq 1/x$, where good 2 is a gross substitute for good 1, see Proposition 2 (B).

[3]As explained in Jensen and Miller (2008), in the calorie-deprived zone the consumer cannot afford to consume any of a fancy good (e.g., meat), in the subsistence zone the consumer responds to an increase in the price of a basic good (e.g., rice or wheat) by reducing consumption of the fancy good in order to fund increased purchases of the basic good, and in the standard zone the consumer responds to an increase in the price of the basic good by reducing consumption of that good.

range of relative prices at which the Giffen behavior appears. The range of income levels is determined so that the equilibrium points exist on the boundaries of (Region-G). We now consider the relative price p^e/q^e and the income level M^e that satisfy the first-order conditions and the budget constraint for a certain equilibrium point $\mathrm{E} = (x^e, y^e)$ that lies in the interior of (Region-G):

$$\frac{U_x}{U_y}(x^e, y^e) = \frac{p^e}{q^e} = \frac{M^e/q^e - y^e}{x^e}. \tag{6}$$

Since good 1 is an inferior good and thus $\partial(U_x/U_y)/\partial y < 0$ holds in (Region-G), the upper and lower bounds of the relative price in the middle of (6) are given by:

$$\frac{U_x}{U_y}\left(x^e, g_u(x^e)\right) < \frac{p^e}{q^e} < \frac{U_x}{U_y}\left(x^e, g_l(x^e)\right). \tag{7}$$

As stated by Vandermeulen (1972, p.454), one implication of the geometric condition for the Giffen behavior is that the real income measured in terms of good 2 that satisfies the equilibrium condition (6),[4]

$$\frac{M}{q}(x, y) = y + x \cdot \frac{U_x}{U_y}(x, y), \tag{8}$$

is monotonically decreasing in y in (Region-G):[5]

$$\frac{\partial}{\partial y}\left(\frac{M}{q}\right)(x, y) = 1 + x\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x, y) < 0.$$

Other implications of the geometric condition presented in this paper are threefold. The first implication is that we can obtain comparative statics results. For the purpose, we calculate the slope of a price-consumption curve (PCC) $M/q = \mathrm{y} + \mathrm{x} \cdot (U_x/U_y)$ (holding q and M fixed) in each region, and then compare it with the slope of an income-

[4]The definition of real income is given in standard textbooks of microeconomic theory. For example, see Jehle and Reny (2011, pp.48-49).

[5]Moffatt and Moffatt (2014) also found this fact by studying necessary and sufficient conditions for the Giffen behavior in the context of a two-good indirect utility function and a reflexion property of Giffen goods. As is readily verified, this condition can also be obtained by applying the implicit function theorem to the first-order conditions.

consumption curve (ICC) $\left(U_x/U_y\right)(x,y) = p/q$ (holding p and q fixed). The results on the slope of the PCC are presented as follows.

Proposition 2: Assume that indifference curves are convex to the origin and thus $(x,y) \in \Lambda_{++} -$ (Region-C), i.e., $-\partial\left(U_x/U_y\right)/\partial y < -\partial\left(U_x/U_y\right)/\partial x \cdot \left(U_y/U_x\right)$.

(A) If $1/x < -\partial\left(U_x/U_y\right)/\partial x \cdot \left(U_y/U_x\right)$, then the PCC is backward-sloping (i.e., the PCC cuts the rotating budget line from above) in (Region-G):

$$\frac{dy}{dx} = -\frac{\partial(M/q)/\partial x}{\partial(M/q)/\partial y} = -\frac{U_x}{U_y}\frac{1/x+\partial\left(U_x/U_y\right)/\partial x\cdot\left(U_y/U_x\right)}{1/x+\partial\left(U_x/U_y\right)/\partial y} < -\frac{U_x}{U_y} < 0,$$

whereas it is upward-sloping $dy/dx > 0$ in (Region-I) and (Region-S).

(B) If $-\partial\left(U_x/U_y\right)/\partial x \cdot \left(U_y/U_x\right) \leq 1/x$, then (Region-G) is empty and the PCC is downward-sloping $0 > dy/dx > -U_x/U_y$ (i.e., the PCC cuts the rotating budget line from right) in (Region-I) and (Region-S).

We then show how the MRS (= the relative price in equilibrium) changes along the PCC by comparing the slopes of the PCC and the ICC.

Proposition 3: Assume that indifference curves are convex to the origin and thus $(x,y) \in \Lambda_{++} -$ (Region-C), i.e., $-\partial\left(U_x/U_y\right)/\partial y \cdot \left(U_x/U_y\right) < -\partial\left(U_x/U_y\right)/\partial x$.

(G) Since $-\partial\left(U_x/U_y\right)/\partial y > 1/x$, the MRS is monotonically increasing in x along the PCC in (Region-G):

$$\frac{d}{dx}\left(\frac{U_x}{U_y}\right)\bigg|_{M/q=const.} = \frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right) + \frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)\cdot\frac{dy}{dx} > 0,$$

where dy/dx is the slope of the PCC presented in Proposition 2.

(I) Since $1/x > -\partial\left(U_x/U_y\right)/\partial y > 0$, the MRS is monotonically decreasing in x along the PCC in (Region-I).

(S) Since $-\partial(U_x/U_y)/\partial y < 0$, the MRS is monotonically decreasing in x along the PCC in (Region-S).

By the convexity of indifference curves and the geometric condition, the slope of the PCC is smaller than the slope of the ICC in (Region-G):[6]

$$\frac{dy}{dx} + \frac{\partial(U_x/U_y)/\partial x}{\partial(U_x/U_y)/\partial y} = \frac{\partial(U_x/U_y)/\partial x - \partial(U_x/U_y)/\partial y \cdot (U_x/U_y)}{x[1/x + \partial(U_x/U_y)/\partial y] \cdot \partial(U_x/U_y)/\partial y} < 0.$$

The desired result (G) follows from this. The other results can be verified in similar ways.

The implications of Propositions 2 and 3 are stated as follows. In (Region-I) of case (A), consumption of good 2 is reduced as the price of good 1 is increased, so that the income share spent on good 1 is monotonically increased along the PCC. Thus, the price elasticity of demand for good 1 is smaller than one in that region. In (Region-G), consumption of good 1 is increased whereas consumption of good 2 is reduced as the price of good 1 is increased, so that the income share spent on good 1 is monotonically increased along the PCC. By contrast, in (Region-I) of case (B), consumption of good 1 is reduced whereas consumption of good 2 is increased as the price of good 1 is increased, so that the income share spent on good 1 is monotonically decreased along the PCC. Thus, the price elasticity of demand for good 1 is greater than one in that region. All the results $dy/dx < -U_x/U_y$ in (Region-G) and $dy/dx > -U_x/U_y$ in (Region-I) and (Region-S) imply that the utility level the consumer can achieve monotonically decreases along the PCC as the price of good 1 is increased.

[6]This result explains the graphical analysis with a figure showing two indifference curves and three budget lines in terms of the properties of utility functions. Consider a situation where an equilibrium point moves from E to E″ due to the total effect of a price reduction (increase) of good 1, and from E to E′ due to the substitution effect, and from E′ to E″ due to the income effect. The convexity of indifference curve implies that the indifference curve through E and E′ cuts the downward-sloping ICC through E′ and E″ from left (right) at E′. The convexity of indifference curve and the geometric condition together imply that the ICC through E′ and E″ cuts the backward-sloping PCC through E and E″ from right (left) at E″.

Remark 1. It can be verified that if good 1 is an inferior good and good 2 is a superior good, then the PCC defined as $M/p = \mathrm{x} + \mathrm{y} \cdot (U_y/U_x)$ (holding p and M fixed) is downward-sloping and the MRS is monotonically decreasing in x along this PCC, meaning that good 1 is a gross substitute for good 2, then $\partial x/\partial q > 0$, in both (Region-G) and (Region-I). Since good 2 is a gross complement to good 1 in case (A) whereas it is a gross substitute for good 1 in case (B), the strong asymmetric gross substitutability occurs in (Region-G) and (Region-I) of case (A), but not in (Region-I) of case (B) (see also footnote 2). It can be said from this and Proposition 2 (B) that if an inferior good and a superior good are mutual substitutes, then the Giffen behavior is impossible. For the necessity of asymmetric gross substitutability for the Giffen behavior, see also de Jaegher (2012, Proposition 5, p.60).

We here present the relation between cases (A) and (B) of Proposition 2.

Proposition 4: Assume that the real income given in (8) is concave in x:

$$\frac{\partial^2}{\partial x^2}\left(\frac{M}{q}\right)(x,y) = 2\frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right)(x,y) + x\frac{\partial^2}{\partial x^2}\left(\frac{U_x}{U_y}\right)(x,y) < 0,$$

where $\partial(U_x/U_y)/\partial x < 0$. Then the region (A) (if any) in which condition (A) of Proposition 2 is satisfied is to the right of the region (B) (if any) in which condition (B) is satisfied.

(Figures 1 and 2 about here)

As a result of Propositions 1, 2, 3, and 4 we can draw the rough shape of the PCC for various utility functions. Figures 1 and 2 respectively show the rough shape of the PCCs for case (a) $-\partial^2(U_x/U_y)/\partial y^2 < 0$ and case (b) $-\partial^2(U_x/U_y)/\partial y^2 > 0$, provided that the concavity assumption of Proposition 4 is satisfied. In Figure 1, (Region-G) is south of (Region-I), and thus the Giffen behavior appears for relatively low levels of income (i.e., the real income is bounded above), relatively high prices of good 1 (i.e., the relative price is bounded below), and relatively high income shares spent on good 1 (i.e., the income share is bounded below). The utility level the consumer can achieve is relatively low (i.e., the utility level is bounded above). By contrast, in Figure 2, (Region-G) is north of (Region-I), and thus the Giffen behavior appears for relatively high levels of income (i.e., the real income is bounded below), relatively low prices of good 1 (i.e., the relative price is bounded above), and relatively low income shares spent on good 1 (i.e., the income share is bounded above). The utility level the consumer can achieve is relatively high (i.e., the utility level is bounded below).

The second implication of the geometric condition is that we can calculate the range of income levels for which equilibrium points exist in (Region-G). Since the real income (8) is monotonically decreasing in y in the interior of (Region-G), the upper and lower bounds of the real income on the right-hand side of (6) are given by:

$$\frac{M}{q}\left(x^e, g_u(x^e)\right) < \frac{M^e}{q^e} < \frac{M}{q}\left(x^e, g_l(x^e)\right) \tag{9}$$

Moreover, (8) is monotonically decreasing in x along the upper and lower boundaries of (Region-G). Specifically, (8) is monotonically decreasing in x along the upper (lower) boundary of (Region-G) in case (a) (case (b)), respectively[7]

[7]In the case of the Moffatt (2012) utility function $U = x + y - \sqrt{\Phi(x,y)}$, where $\Phi(x,y) = [x + y - 2\lambda(1-\lambda)]^2 - 4(1-\lambda^2)[xy - (1-\lambda)^2]$ and $0 < \lambda < 1$, the border between (Region-G) and (Region-I) is a closed curve $2(1-\lambda^2)x\left[(x-y)/\sqrt{\Phi} - 1\right] - \sqrt{\Phi} + 0.5\Phi_y = 0$ with a singular point $((1-\lambda)/\lambda, (1-\lambda)/\lambda)$ where $\Phi = \Phi_x = \Phi_y = 0$, which is also a singular point of an indifference curve. Thus, the condition $-\partial(U_x/U_y)/\partial y = 1/x$ cannot be solved for $y = g_u(x)$ explicitly. However, since the real income (8) is monotonically decreasing in x along the boundary of (Region-G), the upper and lower bounds of the real income are determined numerically. For example, when $\lambda = 0.8$ the

$$\frac{d}{dx}\left(\frac{M}{q}\right)\left(x, g_u(x)\right) = x \cdot \frac{U_x}{U_y}\left[\frac{1}{x} + \frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right) \cdot \frac{U_y}{U_x}\right] < 0.$$

This is because $-\partial(U_x/U_y)/\partial y = 1/x$ holds on the border between (Region-G) and (Region-I) and the value in the bracket is negative when (Region-G) is not empty. Similarly, (8) is monotonically decreasing in x along the lower (upper) boundary of (Region-G) in case (a) (case (b)), respectively

$$\frac{d}{dx}\left(\frac{M}{q}\right)\left(x, g_l(x)\right) = x\left[g_l'(x) + \frac{U_x}{U_y}\right]\left[\frac{1}{x} + \frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right) \cdot \frac{U_y}{U_x}\right] < 0.$$

This is because $-\partial(U_x/U_y)/\partial y = -\partial(U_x/U_y)/\partial x \cdot (U_y/U_x)$ and $g_l'(x) > -U_x/U_y$ hold on the border between (Region-C) and (Region-G).

Thus, if the real income M^e/q^e is smaller than the maximum value of (8) on the upper boundary, then it follows from (9) that there exists an equilibrium point $E_u = \left(x_u^e, g_u(x_u^e)\right)$ with $x_u^e < x^e$ on the upper boundary that satisfies the equilibrium condition:

$$\frac{M^e}{q^e} = \frac{M}{q}\left(x_u^e, g_u(x_u^e)\right) \tag{10}$$

for given $M^e/q^e < max(M/q)\left(x, g_u(x)\right)$.[8] By definition this equilibrium point E_u is a point of intersection of the PCC passing through E and the upper boundary of (Region-G). Similarly, if the real income M^e/q^e is greater than the minimum value of (8) on the lower boundary, then it follows from (9) that there exists an equilibrium point $E_l = \left(x_l^e, g_l(x_l^e)\right)$ with $x^e < x_l^e$ on the lower boundary that satisfies the equilibrium condition:

$$\frac{M^e}{q^e} = \frac{M}{q}\left(x_l^e, g_l(x_l^e)\right) \tag{11}$$

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real income satisfies $0.23 < M^e/q^e < 0.41$, where $0.23 \approx (M/q)(0.3652, 0.1852)$ and $0.41 \approx (M/q)(0.25, 0.2476)$.

[8]In the case where the MRS is zero on the upper boundary of (Region-G), as in Vandermeulen (1972) and Doi et al. (2009), the sufficient conditions for the existence of E_u are written as $g_u'(x) < 0$ and $M^e/q^e < \max g_u(x)$.

for given $\min(M/q)(x, g_l(x)) < M^e/q^e$.[9] By definition this E_l is a point of intersection of the PCC passing through E and the lower boundary of (Region-G).

The third implication of the geometric condition is that we can calculate the range of relative prices. As shown in Proposition 3 (G), the MRS is monotonically increasing in x along the PCC from E_u to E_l. Therefore, the lower bound of the relative price at which the Giffen behavior appears for given q^e and M^e is given by:

$$\frac{U_x}{U_y}\left(x_u^e, g_u(x_u^e)\right) < \frac{p}{q^e}. \tag{12}$$

If the MRS is monotonically decreasing in x along the upper boundary of (Region-G), or equivalently, the upper boundary cuts the downward-sloping ICCs from left, $g_u'(x) > -\partial(U_x/U_y)/\partial x/\partial(U_x/U_y)/\partial y$, then the lower bound given in (12) is greater than the lower bound given in (7):

$$\frac{U_x}{U_y}\left(x^e, g_u(x^e)\right) < \frac{U_x}{U_y}\left(x_u^e, g_u(x_u^e)\right) < \frac{p^e}{q^e}.$$

Similarly, the upper bound of the relative price at which the Giffen behavior appears for given q^e and M^e is given by:

$$\frac{p}{q^e} < \frac{U_x}{U_y}\left(x_l^e, g_l(x_l^e)\right). \tag{13}$$

Since the MRS is monotonically decreasing in x along the border between (Region-C) and (Region-G), i.e., $g_l{}'(x) > -U_x/U_y$ at any point $(x, g_l(x))$, the upper bound given in (13) is smaller than the upper bound given in (7):

$$\frac{p^e}{q^e} < \frac{U_x}{U_y}\left(x_l^e, g_l(x_l^e)\right) < \frac{U_x}{U_y}\left(x^e, g_l(x^e)\right).$$

[9]This means that if the lower bound of (8) on the lower boundary is zero, then the real income M^e/q^e is not bounded below. In the cases of Vandermeulen (1972) and Doi et al. (2009), not only $g_l'(x) < 0$ and $g_l(+\infty) = 0$ but also $g_u'(x) < 0$ and $g_u(0) = +\infty$ hold (see also footnote 8). This is the reason why in their cases the Giffen behavior is independent of the income level.

The difference between the upper bound (13) and the lower bound (12) is sufficiently small, so that the pure substitution effect is smaller than the income effect.

Since the income share spent on good 1 is monotonically increased along the PCC from E_u to E_l, the income share for the Giffen good satisfies

$$\frac{x_u^e \cdot (U_x/U_y)(x_u^e, g_u(x_u^e))}{g_u(x_u^e) + x_u^e \cdot (U_x/U_y)(x_u^e, g_u(x_u^e))} < \frac{px}{M^e} < \frac{x_l^e \cdot (U_x/U_y)\left(x_l^e, g_l(x_l^e)\right)}{g_l(x_l^e) + x_l^e \cdot (U_x/U_y)\left(x_l^e, g_l(x_l^e)\right)} \tag{14}$$

for given q^e and M^e (note that the denominators on both sides of (14) equal M^e/q^e). The implications of (14) are twofold. First, this implies that if the MRS is zero on the upper boundary of (Region-G), i.e., a consumer is rapidly satiated with consumption of good 1, then the income share for that good can be arbitrarily low, as in Vandermeulen (1972) and Doi et al. (2009). Second, (14) also implies that the lower bound of the income share for the Giffen good cannot be zero in the case of additive-separable utility functions because $xU_x/U_y = U_y/U_{yy} > 0$ when $-\partial\left(U_x/U_y\right)/\partial y = 1/x$. Specifically, in that case, if and only if utility provided by consumption of good 2 is represented by a power (CRRA) function $\delta y^\gamma/\gamma$ with $\delta > 0$ and $\gamma > 1$, the lower bound of the income share for the Giffen good is $0 < 1/\gamma < 1$ (see also Section 4).

3.2 The Results for the E-S Utility Function

In what follows, we prove that good 1 is an inferior good but not a Giffen good by using our geometrical method. In the case of the E-S utility function, the set Λ_{++} is given by $\Lambda_{++} \equiv \{(x,y) | \delta(x+y)^{\gamma-1} - \beta x^{\alpha-1} > 0\}$. The CMRS function for this utility function is always positive:

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x,y) = (1-\gamma)\frac{\beta x^{\alpha-1}}{\delta(x+y)^\gamma} > 0,$$

where $\delta > 0$, $0 < \gamma < 1$, $\beta > 0$, and $\alpha > 1$. Since the marginal utility of each good is positive for any $(x, y) \in \Lambda_{++}$, we have

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x, y) < \frac{1-\gamma}{x+y} < \frac{1}{x}.$$

Thus, the set Λ_{++} is a subset of (Region-I) defined in Section 3.1. Therefore, good 1 is an inferior good but not a Giffen good.

(Figure 3 about here)

Figure 3 shows the PCC and budget lines for the parameters values $\delta = 1$, $\gamma = 0.5$, $\beta = 1$, and $\alpha = 2$, and an equilibrium point $(x^e, y^e) = (0.1, 1)$. The price of good 2 and the income level are fixed at $q^e = 1$ and $M^e = 1 + 0.1 \cdot (U_x/U_y)(0.1, 1) \approx 1.089$. The budget lines in Figure 3 correspond to 6 different prices of good 1, namely, p =0, 0.2, 0.4, 0.6, 0.8, and 1. It can be verified from (2) that the MRS is one when $x = 0$ and is zero when $U_x = 0$. The quantity of good 1 demanded is monotonically increased along the PCC as its price is reduced.

We now compare the E-S utility function with the Vandermeulen (1972) utility function. To do so, we consider the utility function of the following form: $U = \phi(x, y) + \varphi(x; \alpha, \beta)$ with $U_x = \phi_x + \varphi' > 0$ and $U_y = \phi_y > 0$. In the case of the E-S utility function where $\phi(x, y) = \delta\,(x + y)^\gamma/\gamma$ with $\delta > 0$ and $0 < \gamma < 1$ and $\varphi(x; \alpha, \beta) = -\beta\, x^\alpha/\alpha$ with $\beta > 0$ and $\alpha > 1$, we find that the interdependency term ϕ of the utility function U has a symmetric property such as $\phi_x = \phi_y = \phi'$ and $\phi_{xx} = \phi_{xy} = \phi_{yy} = \phi''$. By the symmetric property, the CMRS function reduces to

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right) = \frac{\varphi'}{\phi'}\frac{\phi''}{\phi'} > 0. \tag{15}$$

Thus, the utility function does not generate the Giffen behavior for any function φ such that $\phi' + \varphi' > 0$, if $\phi' > 0$ and $0 < -\phi''/\phi' \leq 1/x$. For the E-S utility function the latter condition is satisfied as follows: $0 < -\phi''/\phi' = (1-\gamma)/(x+y) < 1/x$ for any $x > 0$ and $y > 0$ and $0 < \gamma < 1$, which is the inequality used in the proof in Section 2. By the symmetric property, the diminishing MRS condition also reduces to

$$-\frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right)\frac{U_y}{U_x} + \frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right) = -\frac{\varphi''\phi'^2 + \varphi'^2\phi''}{\phi'^2(\phi'+\varphi')} > 0. \tag{16}$$

This condition is satisfied because of $\varphi'' < 0$ and $\phi'' < 0$ for any $x > 0$ and $y > 0$.

On the other hand, in the case of the Vandermeulen (1972) utility function where $\phi(x,y) = x^{-1}y^{\gamma}/\gamma$ with $\gamma > 1$ and $\varphi(x;\alpha,\beta) = -\beta x^{-\alpha}/\alpha$ with $\beta > 0$ and $\alpha > 1$, we can verify that $\phi_x \neq \phi_y$ and $\phi_{xx} \neq \phi_{xy} \neq \phi_{yy}$. The CMRS function is thus written as:

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right) = -\frac{\phi_{xy}}{\phi_y} + U_x\frac{\phi_{yy}}{\phi_y^2},$$

which implies that if good 1 is an inferior good, then $\phi_{xy} < 0$ and/or $\phi_{yy} > 0$. Since the interdependency term satisfies $-\phi_{xy}/\phi_y = 1/x$ and $\phi_{yy} > 0$, the geometric condition for the Giffen behavior is satisfied whenever $U_x > 0$ and thus $x^{\alpha-1}y^{\gamma} < \gamma\beta$. According to Vandermeulen (1972, p.454, footnote 4), the lower boundary of (Region-G) is $x^{\alpha-1}y^{\gamma} = \gamma\beta/\alpha$, when $U = 0$. Thus, the Giffen behavior appears for the relative prices

$$0 < \frac{p}{q^e} < \frac{\alpha-1}{\gamma}\left(\frac{\alpha}{\gamma\beta}\right)^{\frac{1}{\alpha-1}}\left[\frac{\gamma}{\alpha-1+\gamma}\frac{M^e}{q^e}\right]^{\frac{\alpha-1+\gamma}{\alpha-1}}$$

and given q^e and $0 < M^e < +\infty$ (see (10), (11), (12), and (13)). The income share for the Giffen good is $0 < px/M^e < (\alpha-1)/(\alpha-1+\gamma)$ (see (14)). Vandermeulen (1972, p.455) conjectured that with the appropriate choice of income and prices, the share of income spent on good 1 may be chosen at will. However, his conjecture is not true. The income share for the Giffen good is bounded above by the upper bound that is independent of p, q, and M.

It can be verified from (15) and (16) that the utility function $U = \phi(x,y) + \varphi(x;\alpha,\beta)$ that has the symmetric property $\phi_x = \phi_y = \phi'$ and $\phi_{xx} = \phi_{xy} = \phi_{yy} = \phi''$ does not generate the Giffen behavior if $\phi' > 0$ and $0 < -\phi''/\phi' \leq 1/x$ and/or $\varphi' > 0$ and $-x\varphi''/\varphi' \leq 1$ for any $x > 0$ and $y > 0$. The following example shows that a modified E-S utility function with $\phi'' > 0$ and $\varphi(x; \alpha, \beta) = -\beta\, x^{-\alpha}/\alpha$ generates the Giffen behavior.

Example 1. We now consider the following utility function:

$$U = \delta\,\frac{(x+y)^{\gamma}}{\gamma} - \beta\,\frac{x^{-\alpha}}{\alpha},$$

where $\delta > 0$, $\gamma > 1$, $\beta > 0$, and $\alpha > \gamma/(\gamma - 1)$. It is verified that $U_x > 0$ and $U_y > 0$ for any $x > 0$ and $y > 0$ (no satiation). It is also verified that $-x\varphi''/\varphi' = \alpha + 1 > 1$. From (15), the CMRS function for this utility function is always positive, and is monotonically decreasing in y:

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x, y) = (\gamma - 1)\,\frac{\beta x^{-(\alpha+1)}}{\delta(x+y)^{\gamma}} > 0,$$

$$-\frac{\partial^2}{\partial y^2}\left(\frac{U_x}{U_y}\right)(x, y) = -\gamma(\gamma - 1)\,\frac{\beta x^{-(\alpha+1)}}{\delta(x+y)^{\gamma+1}} < 0. \qquad (17)$$

Thus, good 1 is an inferior good and good 2 is a superior good. Since the utility level is positive if $y > -x + (\gamma\beta/\alpha\delta)^{1/\gamma} x^{-\alpha/\gamma}$ (the implied subsistence constraint), (Region-G) is given by

$$G(\alpha, \beta, \gamma, \delta) \equiv \left\{ (x, y) \middle| -x + \left(\frac{\gamma\beta}{\alpha\delta}\right)^{\frac{1}{\gamma}} x^{-\frac{\alpha}{\gamma}} < y < -x + \left[\frac{(\gamma-1)\beta}{\delta}\right]^{\frac{1}{\gamma}} x^{-\frac{\alpha}{\gamma}} \right\}, \qquad (18)$$

where $\gamma/\alpha < \gamma - 1$. (Note that from (16) the diminishing MRS condition is satisfied if $y > -x + [(\gamma - 1)\beta/(\alpha + 1)\delta]^{1/\gamma} x^{-\alpha/\gamma}$, where $(\gamma - 1)/(\alpha + 1) < \gamma/\alpha$.)

Here we identify the ranges of relative prices and income levels for this utility function. To do so, consider an equilibrium point $E = (x^e, y^e)$ that satisfies the equilibrium condition (6) for given prices (p^e, q^e) and income level M^e, and that lies in the interior of (Region-G). From (7) and (18), the upper and lower bounds of the relative price are given by

$$1+\frac{1}{\gamma-1}\left[\frac{(\gamma-1)\beta}{\delta}\right]^{\frac{1}{\gamma}}(x^e)^{-\frac{\alpha+\gamma}{\gamma}}<\frac{p^e}{q^e}<1+\frac{\alpha}{\gamma}\left(\frac{\gamma\beta}{\alpha\delta}\right)^{\frac{1}{\gamma}}(x^e)^{-\frac{\alpha+\gamma}{\gamma}}.$$

It is verified that the upper and lower bounds of the MRS are monotonically decreasing in x. From (9) and (18), the upper and lower bounds of the real income are given by

$$\frac{\gamma}{\gamma-1}\left[\frac{(\gamma-1)\beta}{\delta}\right]^{\frac{1}{\gamma}}(x^e)^{-\frac{\alpha}{\gamma}}<\frac{M^e}{q^e}<\frac{\alpha+\gamma}{\gamma}\left(\frac{\gamma\beta}{\alpha\delta}\right)^{\frac{1}{\gamma}}(x^e)^{-\frac{\alpha}{\gamma}}.$$

It is verified that the upper and lower bounds of the real income are monotonically decreasing in x and that $\min(M/q)(x,g_l(x))=(\alpha+\gamma)/\gamma\cdot(\gamma\beta/\alpha\delta)^{1/(\alpha+\gamma)}$ and $(M/q)(0,g_u(0))=+\infty$ hold. Thus, there exist the equilibrium points $E_l=(x_l^e,g_l(x_l^e))$ with $x^e<x_l^e<(\gamma\beta/\alpha\delta)^{1/(\alpha+\gamma)}$ and $E_u=(x_u^e,g_u(x_u^e))$ with $0<x_u^e<x^e$ on the lower and upper boundaries of (Region-G) respectively, if $(\alpha+\gamma)/\gamma\cdot(\gamma\beta/\alpha\delta)^{1/(\alpha+\gamma)}<M^e/q^e<+\infty$. From (12) and (13), the range of the relative prices at which the Giffen behavior appears is given by

$$1+\frac{1}{\gamma-1}\left[\frac{\delta}{(\gamma-1)\beta}\right]^{\frac{1}{\alpha}}\left[\frac{\gamma-1}{\gamma}\frac{M^e}{q^e}\right]^{\frac{\alpha+\gamma}{\alpha}}<\frac{p}{q^e}<1+\frac{\alpha}{\gamma}\left(\frac{\alpha\delta}{\gamma\beta}\right)^{\frac{1}{\alpha}}\left[\frac{\gamma}{\alpha+\gamma}\frac{M^e}{q^e}\right]^{\frac{\alpha+\gamma}{\alpha}}. \quad (19)$$

From (14), the income share for the Giffen good satisfies

$$\left[\frac{\gamma}{\gamma-1}\right]^{\frac{\gamma}{\alpha}}\left[\frac{(\gamma-1)\beta}{\delta}\right]^{\frac{1}{\alpha}}\left(\frac{q^e}{M^e}\right)^{\frac{\alpha+\gamma}{\alpha}}+\frac{1}{\gamma}<\frac{px}{M^e}<\left[\frac{\alpha+\gamma}{\gamma}\right]^{\frac{\gamma}{\alpha}}\left(\frac{\gamma\beta}{\alpha\delta}\right)^{\frac{1}{\alpha}}\left(\frac{q^e}{M^e}\right)^{\frac{\alpha+\gamma}{\alpha}}+\frac{\alpha}{\alpha+\gamma}.$$

(Figure 4 about here)

Figure 4 shows budget lines and the PCC for the modified E-S utility function with $\gamma=2$, $\beta/\delta=1$, and $\alpha>2$. The PCC in this figure is given by

$$\mathrm{y}=-\mathrm{x}+\frac{M}{2q}\pm\sqrt{\left(\frac{M}{2q}\right)^2-x^{-\alpha}}.$$

The concavity assumption in Proposition 4 is not satisfied for this utility function, and thus the region (A) of Proposition 2 is to the left of the region (B) (change the regions (A) and (B) in Figure 1). From (17) and Proposition 1 (a), (Region-G) is south of (Region-I) in the region (A). The quantity of good 1 demanded is monotonically increased as its price is increased in (Region-G) and the optimal consumption bundle becomes a corner solution if the relative price is higher than the upper bound given on the right-hand side of (19).

Remark 2. The results of Example 1 can be generalized for any $\varphi(x;\alpha,\beta)$ with $\varphi' > 0$, $\varphi'' < 0$, and $-x\varphi''/\varphi' > 1$. If $\varphi(x;\alpha,\beta) < 0$ for any x, as in Example 1, an additional condition $-x\varphi'/\varphi > \gamma/(\gamma-1)$ for $\gamma > 1$ is necessary. For example, the Modified Bergson class of functions studied by Kannai and Selden (2014) satisfy these conditions as follows: The HARA function $\varphi(x;\alpha,\beta) = -(x+\beta)^{-\alpha}/\alpha$ with $\beta > 0$, which is also used in financial economics literature (e.g., Kubler, Selden, and Wei, 2013), satisfies them for $x > \gamma\beta/[(\alpha-1)(\gamma-1)-1]$ and $\alpha > \gamma/(\gamma-1)$. The CARA function $\varphi(x;\alpha,\beta) = -\beta\exp(-\alpha x)/\alpha$ with $\alpha > 0$ and $\beta > 0$ satisfies them for $x > \gamma/\alpha(\gamma-1)$. As is readily verified, the functions $\varphi(x;\alpha,\beta)$ used in Examples 2 (non-CRRA) and 3 (HARA) in Section 4 also satisfy the above conditions for some $(x,y) \in \Lambda_{++}$.

4 THE UTILITY FUNCTIONS THAT GENERATE THE GIFFEN BEHAVIOR

In this section we will apply our geometrical method to the two well-known utility functions, those of Silberberg and Walker (1984) and Spiegel (1994), and then present the ranges of relative prices and income levels at which the Giffen behavior appears and the range of income shares for the Giffen good, which were not satisfactorily explored by preceding studies. To do so, we consider a class of additive-separable utility functions of the following form:

$$\mathrm{U} = \varphi(x; \alpha, \beta) + \delta \frac{y^\gamma}{\gamma},$$

where φ is assumed to be increasing and concave in x, $\varphi' > 0$ and $\varphi'' < 0$, and the parameter values are $\alpha > 0$, $\beta > 0$, $\delta > 0$, and $\gamma > 1$. The MRS for this utility function is

$$-\frac{U_x}{U_y}(x, y) = -\frac{\varphi\prime(x;\alpha,\beta)}{\delta y^{\gamma-1}}. \tag{20}$$

The CMRS function is always positive, and is monotonically decreasing in y:

$$-\frac{\partial}{\partial y}\left(\frac{U_x}{U_y}\right)(x, y) = (\gamma - 1)\frac{\varphi\prime(x;\alpha,\beta)}{\delta y^\gamma} > 0, \tag{21}$$

$$-\frac{\partial^2}{\partial y^2}\left(\frac{U_x}{U_y}\right)(x, y) = -\gamma(\gamma - 1)\frac{\varphi\prime(x;\alpha,\beta)}{\delta y^{\gamma+1}} < 0.$$

Expression (21) implies that good 1 is an inferior good and good 2 is a superior good. Differentiating (20) with respect to x and multiplying the result by the reciprocal of the MRS gives us

$$-\frac{\partial}{\partial x}\left(\frac{U_x}{U_y}\right)(x, y) \cdot \frac{U_y}{U_x}(x, y) = -\frac{\varphi\prime\prime(x;\alpha,\beta)}{\varphi\prime(x;\alpha,\beta)} > 0. \tag{22}$$

The CMRS function (21) is monotonically decreasing in y and takes the values $-\partial(U_x/U_y)/\partial y\,(x, 0) = +\infty$ and $-\partial(U_x/U_y)/\partial y\,(x, +\infty) = 0$. Therefore, the subset Λ_{++} can be divided into three regions, (Region-C), (Region-G), and (Region-I), as shown in Proposition 1 (a). (Region-C) is the set of $(x, y) \in \Lambda_{++}$ such that (21) is greater than (22), and thus it is given by

$$\mathrm{C}(\alpha, \beta, \gamma, \delta) \equiv \left\{(x, y) \middle| 0 < y < \left[-(\gamma - 1)\frac{\left(\varphi\prime(x;\alpha,\beta)\right)^2}{\delta\varphi\prime\prime(x;\alpha,\beta)}\right]^{\frac{1}{\gamma}}\right\}. \tag{23}$$

(Region-G) is the set of $(x, y) \in \Lambda_{++}$ such that (22) is greater than $1/x$ and (21) is between $1/x$ and (22), and thus it is given by

$$\mathrm{G}(\alpha,\beta,\gamma,\delta)\equiv\left\{(x,y)\middle|\left[-(\gamma-1)\frac{(\varphi\prime(x;\alpha,\beta))^2}{\delta\varphi\prime\prime(x;\alpha,\beta)}\right]^{\frac{1}{\gamma}}<y<\left[(\gamma-1)\frac{x\varphi\prime(x;\alpha,\beta)}{\delta}\right]^{\frac{1}{\gamma}}\right\}, \tag{24}$$

where $-x\varphi''/\varphi'>1$. (Region-I) is the set of $(x,y)\in\Lambda_{++}$ such that (21) is between zero and $1/x$, and thus it is given by

$$\mathrm{I}(\alpha,\beta,\gamma,\delta)\equiv\left\{(x,y)\middle|\left[(\gamma-1)\frac{x\varphi\prime(x;\alpha,\beta)}{\delta}\right]^{\frac{1}{\gamma}}<y\right\}. \tag{25}$$

From (20) and (24), we obtain the following equation for the equilibrium point $E_u=\left(x_u^e,g_u(x_u^e)\right)$ located on the upper boundary of (Region-G), which is defined by the equilibrium condition (10) in the previous section

$$x_u^e\cdot\frac{U_x}{U_y}\left(x_u^e,g_u(x_u^e)\right)+g_u(x_u^e)=\frac{\gamma}{\gamma-1}g_u(x_u^e).$$

By definition the left-hand side equals M^e/q^e, and thus the income share spent on each good is respectively given by $p_u^e x_u^e/M^e=1/\gamma$ and $q^e g_u(x_u^e)/M^e=(\gamma-1)/\gamma$.

Remark 3. The results (21) and (24) imply that if either $0<-yU_{yy}/U_y$ for any y or $0<-xU_{xx}/U_x<1$ for any x, then the additive-separable utility function U does not generate the Giffen behavior. It can be verified from (22) that the latter condition is equivalent to $0<-\partial(U_x/U_y)/\partial x\cdot(U_y/U_x)<1/x$, meaning that good 2 is a gross substitute for good 1, then $\partial y/\partial p>0$.[10] The natural relation between the law of demand and risk aversion in the case of additive-separable preferences is known as a special case of the MMP (Milleron, 1974; Mitjuschin and Polterovich, 1978) conditions: the sufficient condition for monotonicity of demand is that the coefficients of relative risk aversion are

[10]Similarly, if $1<-xU_{xx}/U_x$ for any x, then good 2 is a gross complement to good 1. If $-yU_{yy}/U_y<1$, or equivalently, $\partial(U_x/U_y)/\partial y\cdot(U_y/U_x)<1/y$ for any y, then good 1 is a gross substitute for good 2. In the case of additive-separable preferences, the relation between risk aversion and gross substitutability, combined with the relation between gross substitutability and the Giffen behavior (see Remark 1), leads to the relation between risk aversion and the Giffen behavior.

between zero and four for all goods, $0 < -xU_{xx}/U_x < 4$ for any x and $0 < -yU_{yy}/U_y < 4$ for any y (see Mas-Colell, 1991, Section 5).

Example 2. In the case of a generalized Silberberg and Walker (1984) utility function where $\varphi(x;\alpha,\beta) = \alpha \ln x - \beta x$ with $0 \leq x \leq \alpha/\beta$ and $\gamma = 2$, expression (22) is greater than $1/x$ and takes a finite value whenever $0 < x < \alpha/\beta$. On the other hand, the CMRS function (21) is monotonically decreasing in y and takes the values $-\partial(U_x/U_y)/\partial y\,(x,0) = +\infty$ and $-\partial(U_x/U_y)/\partial y\,(x,+\infty) = 0$. Therefore, the subset Λ_{++} can be divided into three regions, (Region-C), (Region-G), and (Region-I), as shown in Proposition 1 (a). It is verified from (23) and (24) and (25) that all the sets change continuously as the parameter β changes: $\lim_{\beta\to 0} C(\alpha,\beta,\delta) = \left\{(x,y)\middle|0 < y < \sqrt{\alpha/\delta}\right\}$, $\lim_{\beta\to 0} G(\alpha,\beta,\delta) = \emptyset$, and $\lim_{\beta\to 0} I(\alpha,\beta,\delta) = \left\{(x,y)\middle|\sqrt{\alpha/\delta} < y\right\}$, which are the results obtained for the Liebhafsky (1969) inferior utility function $U = \alpha \ln x + \delta y^2/2$ with $\alpha > 0$ and $\delta > 0$. The reason why (Region-G) is empty is that in the case of additively (multiplicatively) separable utility function, if and only if utility provided by consumption of good 1 is represented by a logarithmic (CRRA) function $\alpha \ln x$ with $\alpha > 0$ (a power function x^α with $\alpha > 0$), the two critical values equal, $-\partial(U_x/U_y)/\partial x \cdot (U_y/U_x) = 1/x$ for any x. By contrast, (Region-C) cannot be empty for any reasonable parameter values $\alpha > 0$, $0 < \beta < +\infty$, and $\delta > 0$.

Here we identify the ranges of relative prices and income levels for the generalized S-W utility function. To do so, consider an equilibrium point $E = (x^e, y^e)$ that satisfies the equilibrium condition (6) for given prices (p^e, q^e) and income level M^e, and that lies in the interior of (Region-G). From (7) and (24), the upper and lower bounds of the relative price are given by

$$\frac{\sqrt{(\alpha-\beta x^e)/\delta}}{x^e} < \frac{p^e}{q^e} < \frac{\sqrt{\alpha/\delta}}{x^e}. \tag{26}$$

It is verified that the upper and lower bounds of the MRS in (26) are monotonically decreasing in x. From (9) and (24), the upper and lower bounds of the real income are given by

$$2\sqrt{\frac{\alpha-\beta x^e}{\delta}} < \frac{M^e}{q^e} < \frac{2\alpha-\beta x^e}{\sqrt{\alpha\delta}}. \tag{27}$$

It is verified that the lower bound of the real income on the left-hand side of (27) is monotonically decreasing in x and that the maximum value of the lower bound is greater than M^e/q^e. Thus, there exists the equilibrium point $E_u = \left(x_u^e, g_u(x_u^e)\right)$ with $0 < x_u^e < x^e$ that satisfies the equilibrium condition (10) for $M^e/q^e < 2\sqrt{\alpha/\delta}$. From (12), the lower bound of the relative price is given by

$$\frac{2\beta}{\delta}\frac{M^e}{q^e}\left[2\sqrt{\frac{\alpha}{\delta}}+\frac{M^e}{q^e}\right]^{-1}\left[2\sqrt{\frac{\alpha}{\delta}}-\frac{M^e}{q^e}\right]^{-1} < \frac{p}{q^e}. \tag{28}$$

On the other hand, it is verified that the upper bound of the real income on the right-hand side of (27) is monotonically decreasing in x. If the minimum value of the upper bound is smaller than M^e/q^e, then there exists the other equilibrium point $E_l = \left(x_l^e, g_l(x_l^e)\right)$ with $x^e < x_l^e < \alpha/\beta$ that satisfies the equilibrium condition (11) for $\sqrt{\alpha/\delta} < M^e/q^e$. From (13), the upper bound of the relative price is given by[11]

$$\frac{p}{q^e} < \frac{\beta}{\delta}\left[2\sqrt{\frac{\alpha}{\delta}}-\frac{M^e}{q^e}\right]^{-1}. \tag{29}$$

Lastly, we present a necessary condition for the Giffen behavior to be observable. For

[11]Under the parametric assumptions $\alpha = 1$, $\beta = 0.1$, $\delta = 1$, $q^e = 6$, and $M^e = 8.6$, Silberberg and Walker (1984, p.689) showed that the Giffen behavior appears for $p = 1$ and Example 3 of Heijman and von Mouche (2012, pp.74-76) obtained the same result for $p = 0.95$ and $p = 0.9$. Under the same assumptions the range of relative prices (28) and (29) becomes $0.884 < p < 1.059$.

the Giffen behavior to be observable, the utility level the consumer can achieve at the equilibrium point E_u must be higher than the utility level he can achieve at the corner solution $C_u = (M^e/p_u^e, 0)$ that lies on the same budget line as E_u, so that he does not choose C_u for the relative prices satisfying (28) and (29). The following condition must be satisfied:

$$\alpha \ln x_u^e - \beta x_u^e + \delta \frac{g_u(x_u^e)^2}{2} > \alpha \ln\left(\frac{M^e}{p_u^e}\right) - \beta \frac{M^e}{p_u^e}. \tag{30}$$

From (9) and (10) and (27), the equilibrium quantities of goods 1 and 2 on the left-hand side of (30) are given by

$$x_u^e = \frac{\delta}{4\beta}\left[2\sqrt{\frac{\alpha}{\delta}} + \frac{M^e}{q^e}\right]\left[2\sqrt{\frac{\alpha}{\delta}} - \frac{M^e}{q^e}\right] \text{ and } g_u(x_u^e) = \frac{M^e}{2q^e}.$$

From the lower bound of the relative price in (28), the real income measured in terms of good 1 on the right-hand side of (30) is given by

$$\frac{M^e}{p_u^e} = \frac{\delta}{2\beta}\left[2\sqrt{\frac{\alpha}{\delta}} + \frac{M^e}{q^e}\right]\left[2\sqrt{\frac{\alpha}{\delta}} - \frac{M^e}{q^e}\right].$$

Thus, condition (30) is satisfied if the real income measured in terms of good 2 satisfies

$$\frac{M^e}{q^e} < \sqrt{\frac{8(\ln(1/2)+1)\alpha}{\delta}}.$$

Since $1 < \sqrt{8(\ln(1/2)+1)} < 2$, the range of the real incomes at which the Giffen behavior appears is $\sqrt{\alpha/\delta} < M^e/q^e < \sqrt{8(\ln(1/2)+1)\alpha/\delta}$.

(Figure 5 about here)

Figure 5 shows budget lines and the PCC for the generalized S-W utility function. The PCC in this figure is given by

$$y = \frac{1}{2}\sqrt{\frac{\alpha}{\delta}} \pm \frac{1}{2}\sqrt{\frac{4\beta x - 3\alpha}{\delta}}.$$

The value in the square root of the numerator is non-negative and equals zero at the equilibrium point E_u. Figure 5 shows that any consumption bundle that satisfies the equilibrium condition (6) for given $M/q = \sqrt{\alpha/\delta}$ and $0 < p < \beta q/\sqrt{\alpha\delta}$ (see (29)) is a unique interior solution to the constrained utility-maximization problem. (These parameter values were not found by Heijman and von Mouche (2012, pp.74-76).) The income share spent on good 1 is $1/2 < px/M < 1$ when good 1 is a Giffen good whereas it is $0 < px/M < 1/2$ when good 1 is an inferior good. As a result of Propositions 1(a) and 2(A), the PCC is upward-sloping if $M/q \geq 2\sqrt{\alpha/\delta}$ (see also the region (A) of Figure 1).

Example 3. In the case of a modified Spiegel (1994) utility function where $\varphi(x;\alpha,\beta) = \alpha x - \beta x^2/2$ with $0 \leq x \leq \alpha/\beta$ and $\gamma = 2$, expression (22) is greater than $1/x$ and takes a finite value if $\alpha/2\beta < x < \alpha/\beta$, and is smaller than $1/x$ if $0 < x < \alpha/2\beta$.[12] On the other hand, the CMRS function (21) is monotonically decreasing in y and takes the values $-\partial(U_x/U_y)/\partial y\,(x,0) = +\infty$ and $-\partial(U_x/U_y)/\partial y\,(x,+\infty) = 0$. Therefore, the region (A) where $\alpha/2\beta < x < \alpha/\beta$ can be divided into (Region-C), (Region-G), and (Region-I), and the region (B) where $0 < x < \alpha/2\beta$ can be divided into (Region-C) and (Region-I), as shown in Propositions 1(a) and 4.

We now consider an equilibrium point $E = (x^e, y^e)$ that satisfies the equilibrium condition (6) for given prices (p^e, q^e) and income level M^e, and that lies in the interior of (Region-G). From (7) and (24), the upper and lower bounds of the relative price are given by

[12]For the results on the original Spiegel utility function, see Remark 4.

$$\frac{\sqrt{x^e(\alpha-\beta x^e)/\delta}}{x^e} < \frac{p^e}{q^e} < \sqrt{\frac{\beta}{\delta}}. \tag{31}$$

Expression (31) implies that the likelihood for the Giffen phenomenon is greater if p is relatively low and q is relatively high for the small level of β and the high level of δ, as conjectured by Spiegel (1994, p.145). (Comment made by Weber (1997, p.38) is not correct.) From (9) and (24), the upper and lower bounds of the real income are given by

$$2\sqrt{\frac{x^e(\alpha-\beta x^e)}{\delta}} < \frac{M^e}{q^e} < \frac{\alpha}{\sqrt{\beta\delta}}. \tag{32}$$

It is verified that the lower bound of the real income on the left-hand side of (32) is monotonically decreasing in x if $\alpha/2\beta < x < \alpha/\beta$, and that the maximum value of the lower bound is greater than M^e/q^e. Thus, there exists the equilibrium point $E_u = \left(x_u^e, g_u(x_u^e)\right)$ with $\alpha/2\beta < x_u^e < x^e$ that satisfies the equilibrium condition (10) for $M^e/q^e < \alpha/\sqrt{\beta\delta}$. From (12), the lower bound of the relative price is given by

$$\frac{\alpha}{\delta(M^e/q^e)} - \sqrt{\left[\frac{\alpha}{\delta(M^e/q^e)}\right]^2 - \frac{\beta}{\delta}} < \frac{p}{q^e}. \tag{33}$$

The optimal consumption bundle is a corner solution if the relative price is $p/q^e > \beta/\alpha \cdot M^e/q^e$ (the upper bound on the right-hand side is greater than the lower bound given in (33) because of $\alpha - \sqrt{\alpha^2 - \beta\delta(M^e/q^e)^2} > 0$).

(Figure 6 about here)

Figure 6 shows budget lines and the PCC for the modified Spiegel utility function with $\delta = 1$.[13] The PCC in this figure is given by

[13]Figure 1 in Weber (1997, p.38) is not correct even if $\lambda \neq 0$.

$$y = \frac{M}{2q} \pm \sqrt{\beta\left[x - \frac{\alpha}{2\beta}\right]^2 + \frac{1}{4}\left[\frac{M}{q} + \frac{\alpha}{\sqrt{\beta}}\right]\left[\frac{M}{q} - \frac{\alpha}{\sqrt{\beta}}\right]},$$

where $M/q < \alpha/\sqrt{\beta}$. Along the PCC, consumption of good 1 is monotonically reduced as the price of good 1 is reduced in (Region-G) whereas it is monotonically increased as the price of good 1 is reduced in (Region-I). The optimal consumption bundle goes to $(\alpha/\beta, M/q)$ as the price of good 1 falls to zero. The income share spent on good 1 is $1/2 < px/M < 1$ when good 1 is a Giffen good whereas it is $0 < px/M < 1/2$ when good 1 is an inferior good. As a result of Propositions 1 (a) and 2 and 4, the PCC is a U-shaped curve if $M/q > \alpha/\sqrt{\beta}$ (see also Figure 1).

Remark 4. In the original Spiegel utility function, utility provided by consumption of good 2 is given by $\lambda y + \delta y^2/2$ with $\lambda > 0$. If $\lambda > 0$, then the real income must satisfy $\lambda/\delta < M^e/q^e < \alpha/\sqrt{\beta\delta} - \lambda/\delta$ for $\alpha/2\sqrt{\beta\delta} - \lambda/\delta > 0$ so that the Giffen behavior appears for the relative prices at which the quantity demanded of good 2 is positive, $0 < g_u(x_u^e) < g_u(\alpha/2\beta)$, where $g_u(x) = \sqrt{x(\alpha - \beta x)/\delta} - \lambda/\delta$. On the one hand, as shown in Section 3.1, the relative price must be higher than $p_u^e/q^e = (U_x/U_y)(x_u^e, g_u(x_u^e))$ for the Giffen behavior to appear. The lower bound for the case of $\lambda > 0$ is obtained by replacing the term M^e/q^e with $M^e/q^e + \lambda/\delta$ in (32) and (33). On the other hand, the relative price must be lower than $p_0^e/q^e = \alpha/2\lambda - \sqrt{(\alpha/2\lambda)^2 - \beta/\lambda \cdot M^e/q^e}$ for the quantity demanded of good 2 to be positive. The upper bound is obtained by solving $M^e/q^e = (M/q)(x_0^e, 0)$ for x_0^e with $\alpha/2\beta < x_0^e$ and substituting the result into $p_0^e/q^e = (U_x/U_y)(x_0^e, 0)$. It is shown that from Proposition 3 (G) in Section 3.1, the condition $0 < g_u(x_u^e) < g_u(\alpha/2\beta)$ implies $p_u^e < p_0^e$. This is a new and simple proof of the result presented in Example 1 of Heijman and von Mouche (2012, footnote 22, p.82).

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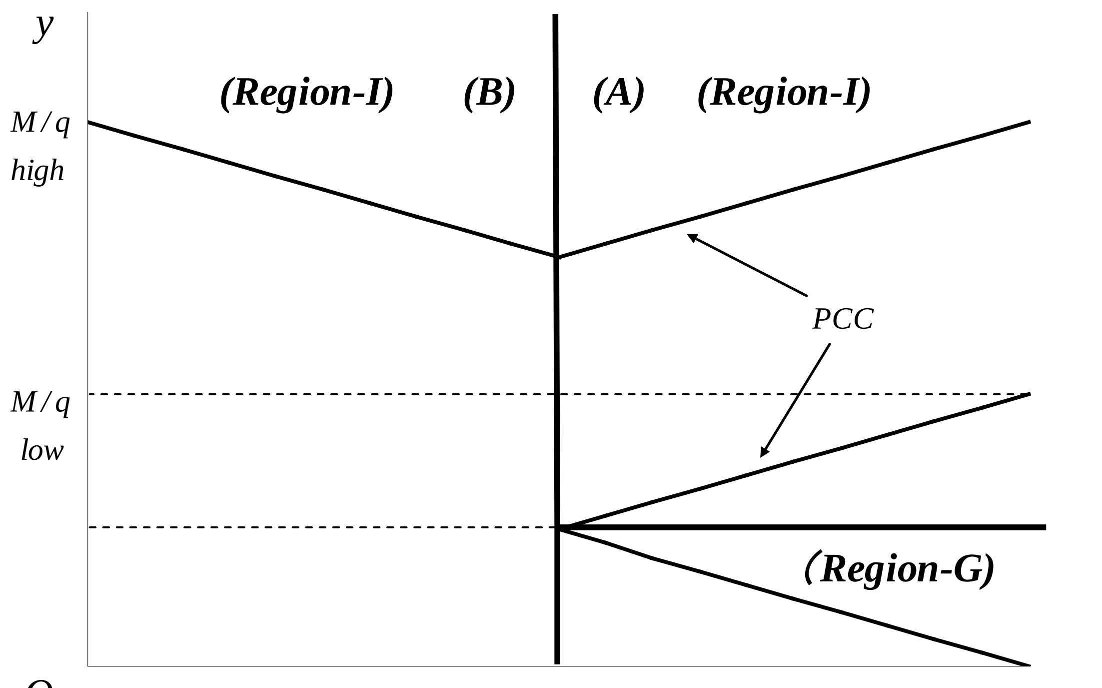

Figure 1. case (a)

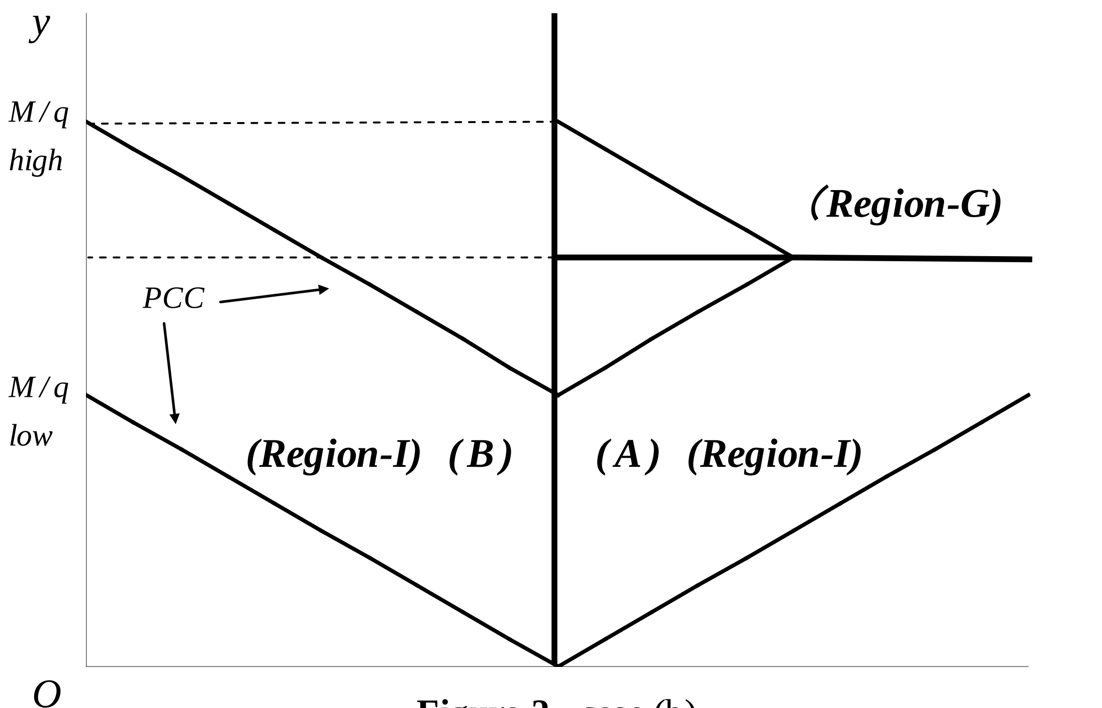

Figure 2. case (b)

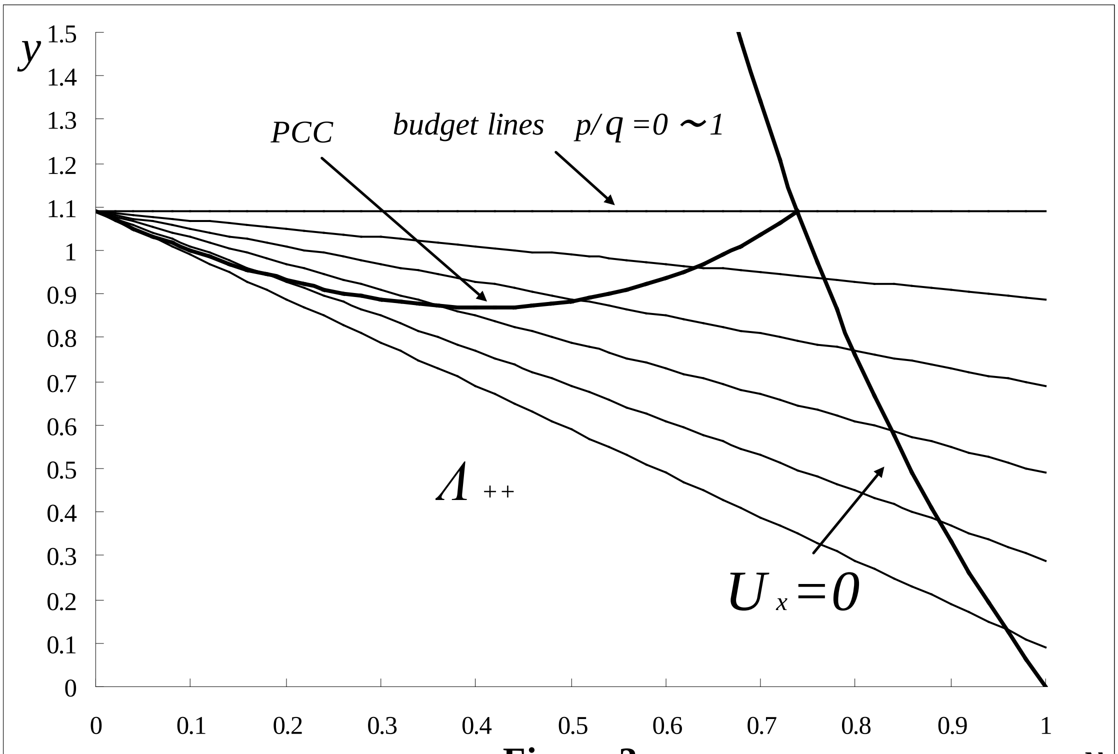

Figure 3.

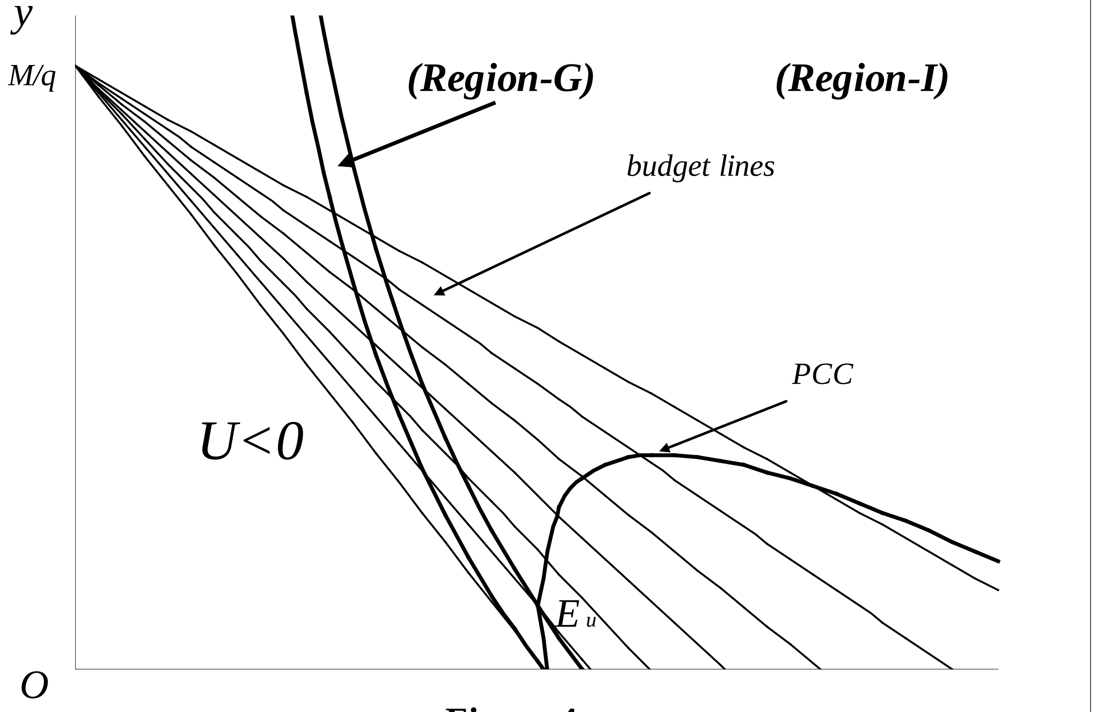

Figure 4.

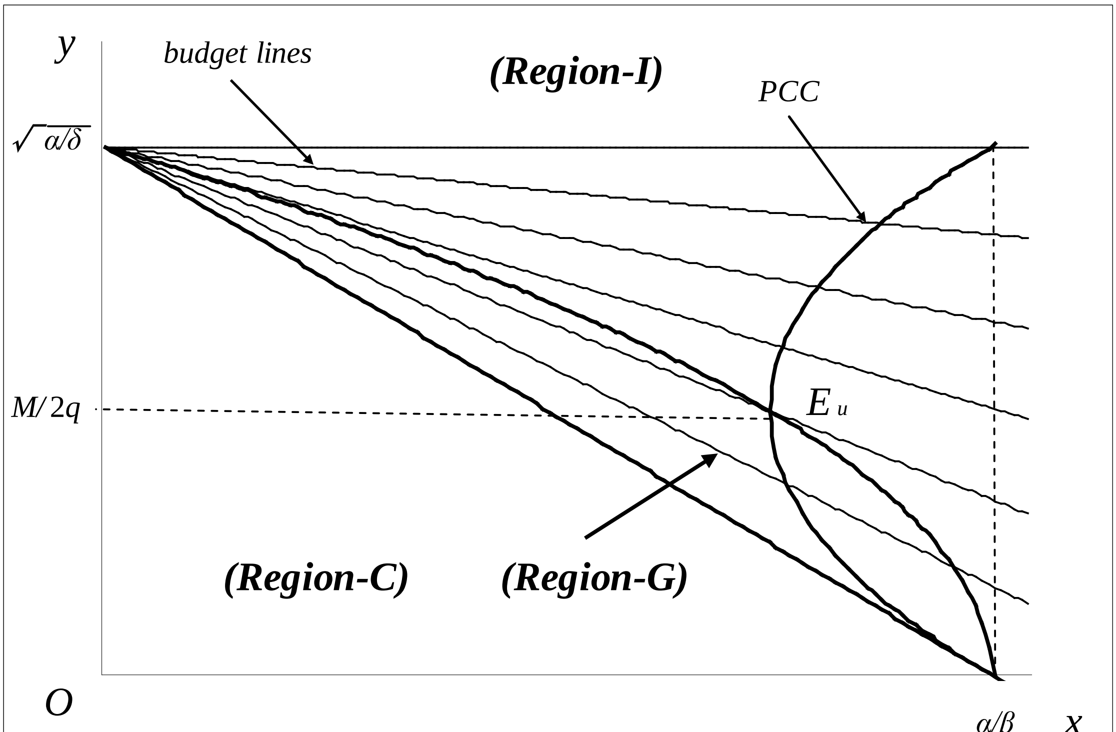

Figure 5.

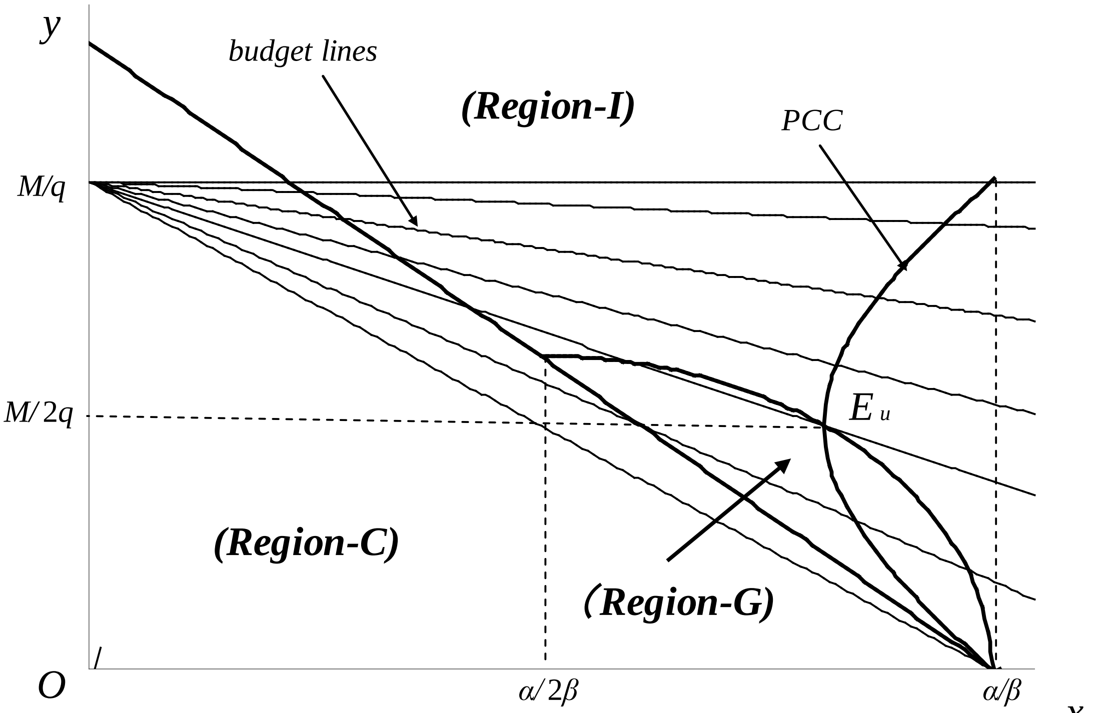

Figure 6.